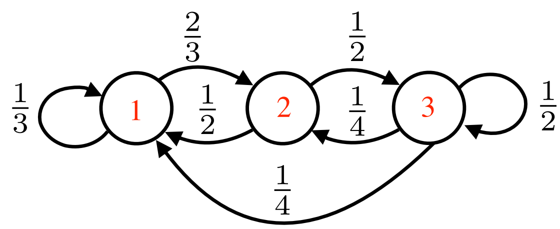
EK381 Probability, Statistics, and Data Science for Engineers — Boston University
Prof Carruthers — Summer 2025

Homework 12: finish by 6/25

Reading: Notes: Chapter 11 **Videos:** 11.1 - 11.4

Problem 12.1 (**Video 11.1 - 11.4**, **Lecture Problem**)

Consider the following Markov chain with initial state probability vector $\underline{p}_0 = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix}$.



(a) Write down the state transition matrix $\mathbf{P}$.

> **Solution:**
>
> The state transition matrix is $\mathbf{P} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$.

(b) Determine the 2-step state transition matrix $\mathbf{P}(2)$. You can use calculators or MATLAB for this computation.

> **Solution:**
>
> The 2-step transition matrix is
>
> $$\mathbf{P}(2) = \mathbf{P}^2 = \mathbf{P} \times \mathbf{P} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 4/9 & 2/9 & 1/3 \\ 7/24 & 11/24 & 1/4 \\ 1/3 & 7/24 & 3/8 \end{bmatrix}.$$

(c) What are the state probability vectors $\underline{p}_1$ and $\underline{p}_2$?

> **Solution:**
>
> $$\underline{p}_1 = \mathbf{P}^T \underline{p}_0 = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}^T \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 11/30 \\ 7/30 \\ 2/5 \end{bmatrix}$$
>
> To determine $\underline{p}_2$, we can either use $\mathbf{P}$ and $\underline{p}_1$ or $\mathbf{P}(2)$ and $\underline{p}_0$. Both possibilities are

shown below:

$$\underline{p}_2 = \mathbf{P}^T \underline{p}_1 = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}^T \begin{bmatrix} 11/30 \\ 7/30 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 61/180 \\ 31/90 \\ 19/60 \end{bmatrix}$$

$$\underline{p}_2 = (\mathbf{P}(2))^T \underline{p}_0 = \begin{bmatrix} 4/9 & 2/9 & 1/3 \\ 7/24 & 11/24 & 1/4 \\ 1/3 & 7/24 & 3/8 \end{bmatrix}^T \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 61/180 \\ 31/90 \\ 19/60 \end{bmatrix}$$

(d) Evaluate $\mathbb{P}[X_0 = 1,\ X_1 = 2,\ X_2 = 2]$ and $\mathbb{P}[X_0 = 3,\ X_1 = 1,\ X_2 = 2]$.

Solution:

Using the multiplication rule,

$$\begin{aligned} \mathbb{P}[X_0 = 1,\ X_1 = 2,\ X_2 = 2] &= \mathbb{P}[X_0 = 1]\,\mathbb{P}[X_1 = 2|X_0 = 1]\,\mathbb{P}[X_2 = 2|X_1 = 2,\ X_0 = 1] \\ &= P[X_0 = 1]\,P[X_1 = 2|X_0 = 1]\,P[X_2 = 2|X_1 = 2] \\ &= p_1(0)\,P_{12}\,P_{22} \\ &= \frac{1}{5} \cdot \frac{2}{3} \cdot 0 \\ &= 0\ . \end{aligned}$$

Intuitively, we can directly infer this is 0 since state 2 cannot transition directly to itself.

Again, using the multiplication rule,

$$\begin{aligned} P[X_0 = 3,\ X_1 = 1,\ X_2 = 2] &= P[X_0 = 3]\,P[X_1 = 1|X_1 = 3]\,P[X_2 = 2|X_1 = 1,\ X_0 = 3] \\ &= P[X_0 = 3]\,P[X_1 = 1|X_0 = 3]\,P[X_2 = 2|X_1 = 1] \\ &= p_3(0)\,P_{31}\,P_{12} \\ &= \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} \\ &= \frac{1}{15}\ . \end{aligned}$$

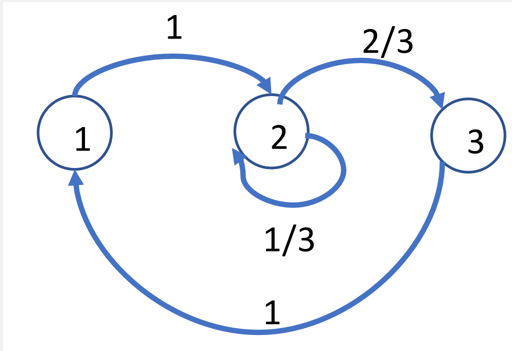
Problem 12.2 (**Video 11.1 - 11.4**)

Consider a Markov chain with the following state transition matrix and initial probability state vector:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/3 & 2/3 \\ 1 & 0 & 0 \end{bmatrix} \qquad \underline{p}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

(a) Draw the Markov chain, labeling the states as 1, 2, and 3, as well as labeling the arcs with the appropriate transition probabilities.

Solution:



(b) What is the period of state 1?

Solution:

There is a cycle of length 3 ($1 \to 2 \to 3 \to 1$) and a cycle of length 4 ($1 \to 2 \to 2 \to 3 \to 1$). The greatest common divisor of those two lengths is 1, so the period of state 1 is 1.

(c) Determine $\mathbb{P}[X_0 = 2, X_1 = 2, X_2 = 3]$.

Solution:

$$\begin{aligned}\mathbb{P}[X_0 = 2, X_1 = 2, X_2 = 3] &= \mathbb{P}[X_2 = 3|X_1 = 2]\,\mathbb{P}[X_1 = 2|X_0 = 2]\,\mathbb{P}[X_0 = 2] \\ &= P_{23}\,P_{22}\,\underline{p}_0(2) \\ &= \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{2}{27}\end{aligned}$$

(d) Does a unique limiting state probability vector $\underline{\pi}$ exist? If so, argue why and solve for it. If not, argue why.

Solution:

This is a recurrent, aperiodic Markov chain. Thus, it has a unique limiting state probability vector $\underline{\pi}$. From the steady-state equation $\mathbf{P}^T\underline{\pi} = \underline{\pi}$,

$$\pi_1 = \pi_3; \quad \frac{2}{3}\pi_2 = \pi_3$$

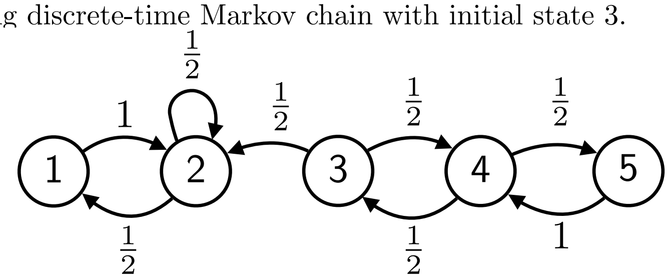
From normalization,

$$\pi_1 + \pi_2 + \pi_3 = \pi_1 + \frac{3}{2}\pi_1 + \pi_1 = \frac{7}{2}\pi_1 = 1$$

So, $\pi_1 = \frac{2}{7}, \pi_2 = \frac{3}{7}, \pi_3 = \frac{2}{7}$ and $\underline{\pi} = \begin{bmatrix} 2/7 \\ 3/7 \\ 2/7 \end{bmatrix}$.

Problem 12.3 (**Video 11.1 - 11.4**, **Lecture Problem**)

Consider the following discrete-time Markov chain with initial state 3.



(a) What are the communicating classes?

> **Solution:**
>
> $$C_1 = \{1, 2\}, \quad C_2 = \{3, 4, 5\}$$

(b) For each communicating class, determine the period and whether it is transient or recurrent.

> **Solution:**
>
> C_1 has period 1, and is recurrent. C_2 has period 2 and is transient.

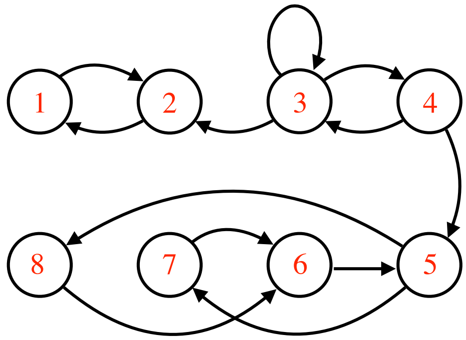
(c) Write down the state transition matrix $\mathbf{P}$.

> **Solution:**
>
> $$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(d) Does a unique limiting state probability vector $\underline{\pi}$ exist? If so, argue why and solve for it. If not, argue why.

> **Solution:**
>
> This Markov chain has one recurrent, aperiodic communicating class while the rest of the states are transient. Thus, it has a unique limiting state probability vector $\underline{\pi}$ where the transient states have $\pi_3 = \pi_4 = \pi_5 = 0$. From the steady-state equation $\mathbf{P}^T \underline{\pi} = \underline{\pi}$, we see that $\frac{1}{2}\pi_2 = \pi_1$ From normalization, $\pi_1 + \pi_2 = 1$ so $\pi_2 = 2/3$ and $\pi_1 = 1/3$.
>
> $$\underline{\pi} = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 12.4 (**Video 11.3**) Consider the following Markov chain. For each of the parts below, you only need to know that each arc represents a positive probability.



(a) Determine the communicating classes.

Solution:

The communicating classes are $C_1 = \{1, 2\}$, $C_2 = \{3, 4\}$, and $C_3 = \{5, 6, 7, 8\}$.

(b) Determine the period for each communicating classes.

Solution:

C_1 has period 2, C_2 has period 1 (and is therefore aperiodic), and C_3 has period 3.

(c) Determine which communicating classes are recurrent and which ones are transient.

Solution:

C_1 and C_3 are recurrent and C_2 is transient.

Problem 12.5 (**Video 11.1 - 11.4**)

Consider a 4 state Markov chain with the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.1 & 0.9 \end{pmatrix}$$

(a) Draw the state transition diagram, with the probabilities for the transitions.

Solution:

Imagine 4 circles, numbered 1 to 4. Arcs from 1 to 1, 1 to 2, 1 to 3, 1 to 4. Arcs from 2 to 2, 2 to 3, 2 to 4. Arcs from 3 to 3, 3 to 4. Arcs from 4 to 4, 4 to 3. Assign the appropriate probabilities from the matrix above.

(b) Find the transient states and recurrent states.

Solution:

States 1 and 2 are transient, as once the chain enters into states 3 or 4, it can never return to states 1 or 2. States 3 and 4 are recurrent.

(c) Is the Markov chain irreducible? Explain.

Solution:

No. It has transient states.

(d) Is the Markov chain aperiodic? Explain.

Solution:

Yes, because there are self-cycles in states 1, 3 and 4.

(e) Find the steady state distribution of this Markov chain.

Solution:

The steady state distribution is nonzero only in states 3 and 4, the recurrent states. By balance out of state 3, we have

$$0.5\pi_3 = 0.1\pi_4$$

so $\pi_4 = 5\pi_3$, so the steady state distribution is $\pi_4 = 5/6, \pi_3 = 1/6$.