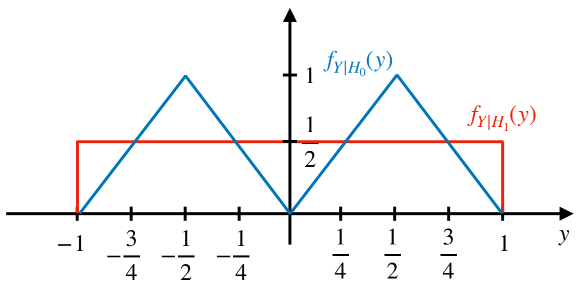
Homework 8: finish by 6/17

Reading: Notes: Chapter 6 **Videos:** 6.1 - 6.4

Problem 8.1 (**Video 6.1, 6.2**, **Lecture Problem**)

Consider the following binary hypothesis testing scenario. (Note that all required integrals can be solved by calculating the areas of rectangles and triangles, so we are expecting exact answers.)



The hypothesis probabilities are $\mathbb{P}[H_0] = 2/3$ and $\mathbb{P}[H_1] = 1/3$.

(a) Determine the ML rule.

Solution:

Looking at the diagram, we can see that $f_{Y|H_0}(y) > f_{Y|H_1}(y)$ whenever $1/4 < |y| < 3/4$. Therefore, the ML rule is

$$D^{\text{ML}}(y) = \begin{cases} 1 & f_{Y|H_1}(y) \geq f_{Y|H_0}(y) \\ 0 & f_{Y|H_1}(y) < f_{Y|H_0}(y) \end{cases}$$
$$= \begin{cases} 1 & 3/4 \leq y \leq 1 \text{ or } -1/4 \leq y \leq 1/4 \text{ or } -1 \leq y \leq -3/4 \\ 0 & -3/4 < y < -1/4 \text{ or } 1/4 < y < 3/4 \end{cases}$$

(b) Determine the MAP rule.

Solution:

The threshold for the likelihood ratio is $\frac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} = 2$. We can see that the likelihood ratio is less than 2 if $1/8 < |y| < 7/8$. Therefore, the MAP rule is

$$D^{\text{MAP}}(y) = \begin{cases} 1 & f_{Y|H_1}(y)\mathbb{P}[H_1] \geq f_{Y|H_0}(y)\mathbb{P}[H_0] \\ 0 & f_{Y|H_1}(y)\mathbb{P}[H_1] < f_{Y|H_0}(y)\mathbb{P}[H_0] \end{cases}$$

$$= \begin{cases} 1 & 7/8 \le y \le 1 \text{ or } -1/8 \le y \le 1/8 \text{ or } -1 \le y \le -7/8 \\ 0 & -7/8 < y < -1/8 \text{ or } 1/8 < y < 7/8 \end{cases}$$

(c) Determine the probability of error under the ML rule.

Solution:

The probability of false alarm is the area of four triangles, with base 1/4 and height 1/2. The probability of missed detection is the area of two rectangles, each of which has base 1/2, height 1/2.

$$\begin{aligned} P_{\text{FA}} &= \mathbb{P}[D^{\text{ML}}(Y) = 1|H_0] = 4 \cdot (1/16) = \frac{1}{4} \\ P_{\text{MD}} &= \mathbb{P}[D^{\text{ML}}(Y) = 0|H_1] = \frac{1}{2} \\ \mathbb{P}[\text{error}_{\text{ML}}] &= P_{\text{FA}}\mathbb{P}[H_0] + P_{\text{MD}}\mathbb{P}[H_1] = \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}. \end{aligned}$$

(d) Determine the probability of error under the MAP rule.

Solution:

The probability of false alarm is the area of four triangles, with base 1/8 and height 1/4. The probability of missed detection is the area of two rectangles, each of which has base 3/4, height 1/2.

$$\begin{aligned} P_{\text{FA}} &= \mathbb{P}[D^{\text{MAP}}(Y) = 1|H_0] = \frac{1}{16} \\ P_{\text{MD}} &= \mathbb{P}[D^{\text{MAP}}(Y) = 0|H_1] = \frac{3}{4} \\ \mathbb{P}[\text{error}_{\text{MAP}}] &= P_{\text{FA}}\mathbb{P}[H_0] + P_{\text{MD}}\mathbb{P}[H_1] = \frac{1}{16} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{1}{3} = \frac{7}{24} \end{aligned}$$

Problem 8.2 (**Video 6.1, 6.2, 6.3**, **Quick Calculations**) For each of the scenarios below, determine the requested quantities.

(a) Under H_0, Y is Gaussian$(-1, 1)$. Under H_1, Y is Gaussian$(+1, 1)$. Let $\mathbb{P}[H_0] = 1/3$ and $\mathbb{P}[H_1] = 2/3$. Determine the ML and MAP decision rules.

Solution:

From Video 6.2, we know that the ML rule for this scenario is

$$D^{\text{ML}}(y) = \begin{cases} 1 & y \ge 0 \\ 0 & y < 0 \end{cases}$$

and that the MAP rule is

$$D^{\text{MAP}}(y) = \begin{cases} 1 & y \ge \beta \\ 0 & y < \beta \end{cases}$$

where $\beta = \frac{1}{2}\ln\left(\frac{1/3}{2/3}\right) = -\frac{1}{2}\ln(2)$.

(b) Under H_0, Y is Exponential(1). Under H_1, Y is Exponential(2). Let $\mathbb{P}[H_0] = 1/2$ and $\mathbb{P}[H_1] = 1/2$. Determine the likelihood ratio, the ML rule, and the probability of error under the ML rule.

Solution:

We start by writing the likelihoods

$$f_{Y|H_0} = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases} \qquad \begin{cases} 2e^{-2y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

and the likelihood ratio as well as the log-likelihood ratio

$$L(y) = \frac{f_{Y|H_1}(y)}{f_{Y|H_0}(y)} = 2e^{-y} \qquad \ln(L(y)) = -y + \ln(2)$$

It follows that

$$D^{\text{ML}}(y) = \begin{cases} 1 & \ln(L(y)) \geq 0 \\ 0 & \ln(L(y)) < 0 \end{cases} = \begin{cases} 1 & -y + \ln(2) \geq 0 \\ 0 & -y + \ln(2) < 0 \end{cases} = \begin{cases} 1 & y \leq \ln(2) \\ 0 & y > \ln(2) \end{cases}$$

The probabilities of false alarm and missed detection are

$$P_{\text{FA}} = \mathbb{P}[Y \leq \ln(2)|H_0] = F_{Y|H_0}(\ln(2)) = 1 - e^{-\ln(2)} = 1 - \frac{1}{2} = \frac{1}{2}$$
$$P_{\text{MD}} = \mathbb{P}[Y > \ln(2)|H_1] = 1 - F_{Y|H_1}(\ln(2)) = e^{-2\ln(2)} = \frac{1}{4}$$

It follows that the probability of error is $P_{\text{error}} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$.

(c) Under H_0, Y is Binomial$(4, 1/2)$. Under H_1, Y is Binomial$(3, 1/2)$. Let $\mathbb{P}[H_0] = 2/3$ and $\mathbb{P}[H_1] = 1/3$. Determine the probability of error under the ML and MAP decision rules.

Solution:

The maximum likelihood (ML) decision rule assigns each outcome x to the hypothesis, H_0 or H_1, with the higher likelihood, $P_{Y|H_0}(y)$ or $P_{Y|H_1}(y)$. We first write out the likelihood table, using the conditional PMFs

$$P_{Y|H_0}(y) = \begin{cases} \binom{n}{y}\left(\frac{1}{2}\right)^4 & y = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases} \qquad P_{Y|H_1}(y) = \begin{cases} \binom{n}{y}\left(\frac{1}{2}\right)^3 & y = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

and then pick the largest element in each column (shown in blue below):

	x				
	0	1	2	3	4
$P_{Y\|H_0}(y)$	1/16	4/16	6/16	4/16	1/16
$P_{Y\|H_1}(y)$	2/16	6/16	6/16	2/16	0

The green-colored elements indicate a tie. Where there is a tie, the ML rule is ambiguous. In this case, the lower probability of error is obtained if we decide the apriori more probable hypothesis, which is H_0. If the hypotheses are equally likely (not the case in this problem), then deciding H_0 or H_1 would give you the same error probability.

Therefore, the ML rule is $D^{\mathrm{ML}}(y) = \begin{cases} 1 & y = 0, 1 \\ 0 & y = 2, 3, 4 \end{cases}$, i.e. we have decision regions $A_1^{\mathrm{ML}} = \{0, 1\}$ and $A_0^{\mathrm{ML}} = \{2, 3, 4\}$. We can now work out the probabilities of false alarm, missed detection, and error

$$P_{\mathrm{FA}}^{\mathrm{ML}} = \sum_{y \in A_1^{\mathrm{ML}}} P_{Y|H_0}(y) = P_{Y|H_0}(0) + P_{Y|H_0}(1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$P_{\mathrm{MD}}^{\mathrm{ML}} = \sum_{y \in A_0^{\mathrm{ML}}} P_{Y|H_1}(y) = P_{Y|H_1}(2) + P_{Y|H_1}(3) + P_{Y|H_1}(4) = \frac{6}{16} + \frac{2}{16} + 0 = \frac{8}{16}$$

$$P_{\mathrm{error}}^{\mathrm{ML}} = P_{\mathrm{FA}}^{\mathrm{ML}}\, \mathbb{P}[H_0] + P_{\mathrm{MD}}^{\mathrm{ML}}\, \mathbb{P}[H_1] = \frac{5}{16} \cdot \frac{2}{3} + \frac{8}{16} \cdot \frac{1}{3} = \frac{3}{8}$$

For completeness, we will work out everything if we had decided H_1 when $x = 2$ (the likelihood ratio is 1):

In this case, the ML rule is given by $D^{\mathrm{ML}}(y) = \begin{cases} 1 & y = 0, 1, 2 \\ 0 & y = 3, 4 \end{cases}$, i.e. we have decision regions $A_1^{\mathrm{ML}} = \{0, 1, 2\}$ and $A_0^{\mathrm{ML}} = \{3, 4\}$. We can now work out the probabilities of false alarm, missed detection, and error

$$P_{\mathrm{FA}}^{\mathrm{ML}} = \sum_{y \in A_1^{\mathrm{ML}}} P_{Y|H_0}(y) = P_{Y|H_0}(0) + P_{Y|H_0}(1) + P_{Y|H_0}(2) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$$

$$P_{\mathrm{MD}}^{\mathrm{ML}} = \sum_{y \in A_0^{\mathrm{ML}}} P_{Y|H_1}(y) = P_{Y|H_1}(3) + P_{Y|H_1}(4) = \frac{2}{16} + 0 = \frac{2}{16}$$

$$P_{\mathrm{error}}^{\mathrm{ML}} = P_{\mathrm{FA}}^{\mathrm{ML}}\, \mathbb{P}[H_0] + P_{\mathrm{MD}}^{\mathrm{ML}}\, \mathbb{P}[H_1] = \frac{11}{16} \cdot \frac{2}{3} + \frac{2}{16} \cdot \frac{1}{3} = \frac{4}{8}$$

which is higher than 3/8 that we got previously when we decided H_0 for $x = 2$.

To determine the MAP rule, we should first scale the rows of the table by the corresponding hypothesis probabilities and then choose the largest element in each column (shown in blue below):

	x				
	0	1	2	3	4
$P_{Y\|H_0}(y)\,\mathbb{P}[H_0]$	1/24	1/6	1/4	1/6	1/24
$P_{Y\|H_1}(y)\,\mathbb{P}[H_1]$	1/24	1/8	1/8	1/24	0

Once again, we notice that the hypotheses H_0 and H_1 are tied when $x = 0$. But unlike the ML rule, the MAP rule is the rule which minimizes the probability of error. Because of this, it turns out that the error probability will be the same irrespective of how we decide when $x = 0$. To demonstrate this, let us first decide H_1 when $x = 0$. In this case, the MAP rule becomes: $D^{\text{MAP}}(y) = \begin{cases} 1 & y = 0 \\ 0 & y = 1, 2, 3, 4 \end{cases}$, i.e. we have decision regions $A_1^{\text{MAP}} = \{0\}$ and $A_0^{\text{MAP}} = \{1, 2, 3, 4\}$. We can now work out the probabilities of false alarm, missed detection, and error

$$\begin{aligned}
P_{\text{FA}}^{\text{MAP}} &= \sum_{y \in A_1^{\text{MAP}}} P_{Y|H_0}(y) = P_{Y|H_0}(0) = \frac{1}{16} \\
P_{\text{MD}}^{\text{MAP}} &= \sum_{y \in A_0^{\text{MAP}}} P_{Y|H_1}(y) \\
&= P_{Y|H_1}(1) + P_{Y|H_1}(2) + P_{Y|H_1}(3) + P_{Y|H_1}(4) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} + 0 = \frac{7}{8} \\
P_{\text{error}}^{\text{MAP}} &= P_{\text{FA}}^{\text{MAP}}\,\mathbb{P}[H_0] + P_{\text{MD}}^{\text{MAP}}\,\mathbb{P}[H_1] = \frac{1}{16} \cdot \frac{2}{3} + \frac{7}{8} \cdot \frac{1}{3} = \frac{1}{3}
\end{aligned}$$

Now suppose that we decide H_0 (instead of H_1) when $x = 0$. In this case, the MAP rule becomes: $D^{\text{MAP}}(y) = \begin{cases} 1 & \text{never} \\ 0 & y = 0, 1, 2, 3, 4 \end{cases}$, i.e. we have decision regions $A_1^{\text{MAP}} = \{\ \}$ and $A_0^{\text{MAP}} = \{0, 1, 2, 3, 4\}$. We can now work out the probabilities of false alarm, missed detection, and error

$$\begin{aligned}
P_{\text{FA}}^{\text{MAP}} &= \sum_{y \in A_1^{\text{MAP}}} P_{Y|H_0}(y) = 0 \\
P_{\text{MD}}^{\text{MAP}} &= \sum_{y \in A_0^{\text{MAP}}} P_{Y|H_1}(y) \\
&= P_{Y|H_1}(0) + P_{Y|H_1}(1) + P_{Y|H_1}(2) + P_{Y|H_1}(3) + P_{Y|H_1}(4) = \frac{2}{16} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} + 0 = 1 \\
P_{\text{error}}^{\text{MAP}} &= P_{\text{FA}}^{\text{MAP}}\,\mathbb{P}[H_0] + P_{\text{MD}}^{\text{MAP}}\,\mathbb{P}[H_1] = 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}
\end{aligned}$$

which is the same error probability we got when we decide H_1 for $x = 0$.

Problem 8.3 (**Video 6.1, 6.2, 6.3**)

This problem is meant to walk through some of the steps carried out algorithmically in Lab 8. (It will likely be easier to complete this problem once you have finished at least the first half of the lab.) Below are tables of training and testing data. You will use the training data to determine the Gaussian ML rule and then use the testing data to evaluation its performance.

Table 1: Training Data

Y	label
3.2	1
0.8	1
0.1	0
-2.1	0

Table 2: Testing Data

Y	label
2.0	1
0.7	1
0.1	1
1.1	0
-2.9	0
-1.6	0

(a) Use the training data to estimate the mean μ_0 under label 0 and the mean μ_1 under label 1. This can be done by simply averaging the training with that label.

Solution:

$$\mu_0 = \frac{1}{2}(0.1 - 2.1) = -1 \qquad \mu_1 = \frac{1}{2}(3.2 + 0.8) = 2$$

(b) Assuming the variances under label 0 and 1 are equal to σ^2, determine the ML rule $D_{\text{ML}}(y)$. Specifically, assume that the data Y with label 0 is generated according to Gaussian(μ_0, σ^2) and the data Y with label 1 is generated according to Gaussian(μ_1, σ^2).

Solution:

For Gaussians with equal variance, the ML rule takes a simple form, which we can quickly derive using the log-likelihood ratio:

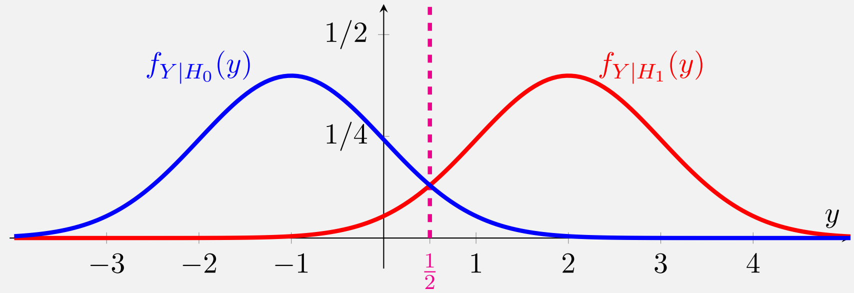
$$\begin{aligned}
L(y) &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(y-\mu_1)^2}{2\sigma^2} \right)}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(y-\mu_0)^2}{2\sigma^2} \right)} = \exp\left(\frac{(y-\mu_0)^2}{2\sigma^2} - \frac{(y-\mu_1)^2}{2\sigma^2} \right) \\
\ln(L(y)) &= \frac{(y-\mu_0)^2}{2\sigma^2} - \frac{(y-\mu_1)^2}{2\sigma^2} \\
D_{\text{ML}}(y) &= \begin{cases} 1 & \ln(L(y)) \geq 0 \\ 0 & \ln(L(y)) < 0 \end{cases} = \begin{cases} 1 & (y-\mu_0)^2 - (y-\mu_1)^2 \geq 0 \\ 0 & (y-\mu_0)^2 - (y-\mu_1)^2 < 0 \end{cases}
\end{aligned}$$

To simplify this further, we need to plug in our values $\mu_0 = -1$ and $\mu_1 = 2$ in to get $(y+1)^2 - (y-2)^2 = y^2 + 2y + 1 - y^2 + 4y - 4 = 6y - 3$. We finally get that

$$D_{\text{ML}}(y) = \begin{cases} 1 & 6y - 3 \geq 0 \\ 0 & 6y - 3 < 0 \end{cases} = \begin{cases} 1 & y \geq \frac{1}{2} \\ 0 & y < \frac{1}{2} \end{cases}$$

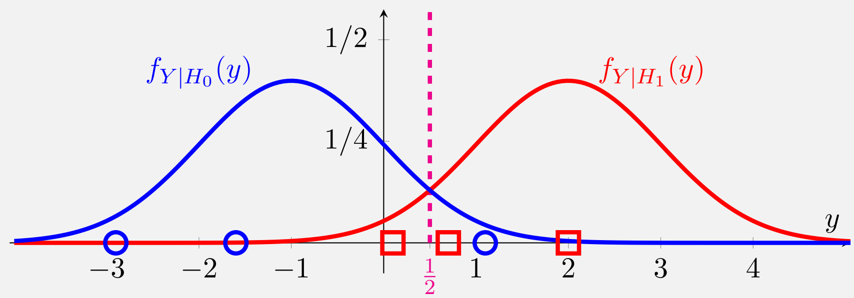
(c) Sketch the conditional PDFs Gaussian(μ_0, σ^2) and Gaussian(μ_1, σ^2) on the same plot along with the decision boundary from the ML rule. (You can assume $\sigma^2 = 1$.)

Solution:



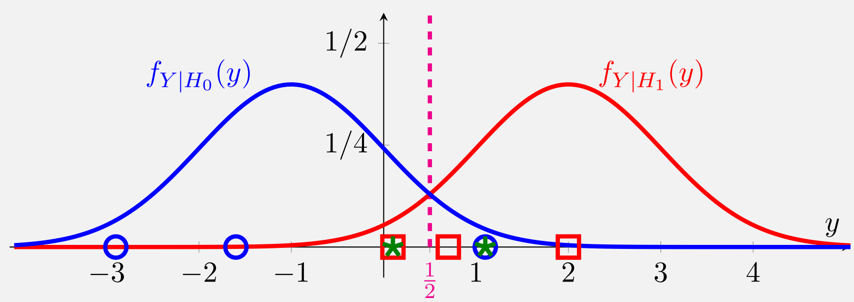
(d) Add the testing data to your sketch just as in Lab 8. Use circles for testing data with label 0 and squares for testing data with label 1.

Solution:



(e) Draw a star inside each testing data on the plot that will be misclassified by the ML rule.

Solution:



(f) Estimate the probability of error as the fraction of misclassified points.

Solution:

2 out of 6 test data points are misclassified so we estimate the probability of error as $2/6 = 1/3$.

Problem 8.4 (**Video 7.1, 7.2**, **Lecture Problem**)

Consider the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{4}{\pi} & x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note this is a uniform distribution over a quarter disk of radius 1.

(a) Determine the MMSE estimator $\hat{x}_{\mathrm{MMSE}}(y)$ of X given $Y = y$.

Solution:

To solve for $\hat{x}_{\mathrm{MMSE}}(y) = \mathbb{E}[X|Y = y]$, we will first need the conditional PDF $f_{X|Y}(x|y)$, which itself requires us to find the marginal PDF $f_Y(y)$.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^{\sqrt{1-y^2}} \frac{4}{\pi}\,dx & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{4}{\pi}\sqrt{1-y^2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) > 0 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{\sqrt{1-y^2}} & x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{x}_{\mathrm{MMSE}}(y) = \mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_0^{\sqrt{1-y^2}} \frac{x}{\sqrt{1-y^2}}\,dx = \frac{1}{2}\sqrt{1-y^2}$$

(b) Determine the Mean Square Error of the MMSE estimator $\mathbb{E}\big[\big(X - \hat{x}_{\mathrm{MMSE}}(Y)\big)^2\big]$.

Solution:

Plugging in the specific form of the MMSE estimator, we get

$$\begin{aligned} \mathbb{E}\big[\big(X - \hat{x}_{\mathrm{MMSE}}(Y)\big)^2\big] &= \mathbb{E}\left[\left(X - \frac{1}{2}\sqrt{1-Y^2}\right)^2\right] \\ &= \mathbb{E}\left[X^2 - X\sqrt{1-Y^2} + \frac{1}{4}(1-Y^2)\right] \\ &= \mathbb{E}[X^2] - \mathbb{E}\big[X\sqrt{1-Y^2}\big] + \frac{1}{4} - \frac{1}{4}\mathbb{E}[Y^2]\ . \end{aligned}$$

We now evaluate the individual terms (the numerical answers were obtained via Wolfram Alpha, but could also be derived using trigonometric identities),

$$\begin{aligned} \mathbb{E}[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^1 y^2 \frac{4}{\pi}\sqrt{1-y^2}\,dy = \frac{1}{4} \\ \mathbb{E}[X^2] &= \frac{1}{4} \quad \text{(by symmetry)} \\ \mathbb{E}\big[X\sqrt{1-Y^2}\big] &= \int_{-\infty}^{\infty} x\sqrt{1-y^2} f_{X,Y}(x,y)\,dxdy = \int_0^1 \int_0^{\sqrt{1-y^2}} x\sqrt{1-y^2}\frac{4}{\pi}\,dx\,dy = \frac{3}{8}\ . \end{aligned}$$

Plugging in, we get

$$\mathbb{E}\big[\big(X - \hat{x}_{\mathrm{MMSE}}(Y)\big)^2\big] = \frac{1}{4} - \frac{3}{8} + \frac{1}{4} - \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}\ .$$

(c) Let X, Y be joint Gaussian random variables, with zero mean, and $\mathsf{Var}[X] = \mathsf{Var}[Y] = 2$, $\mathsf{Cov}[X,Y] = 1$. Determine the MMSE estimator $\hat{x}_{\mathrm{MMSE}}(y)$ of X given $Y = y$.

Solution:

For joint Gaussians,

$$\mathbb{E}[X|Y=y] = \mathbb{E}[X] + \frac{\mathsf{Cov}[X,Y]}{\mathsf{Var}[Y]}(y - \mathbb{E}[Y]) = \frac{1}{2}y\ .$$

(d) Determine the Mean Square Error of the MMSE estimator in the previous part, $\mathbb{E}\big[\big(X - \hat{x}_{\text{MMSE}}(Y)\big)^2\big]$.

Solution:

For joint Gaussians,

$$\text{MSE} = \mathsf{Var}[X] - \frac{(\mathsf{Cov}[X,Y])^2}{\mathsf{Var}[Y]} = 2 - \frac{1}{2} = \frac{3}{2}\ .$$