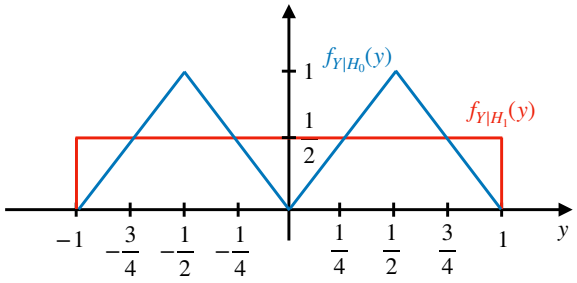
Homework 8: finish by 6/17

Reading: Notes: Chapter 6 **Videos:** 6.1 - 6.4

Problem 8.1 (**Video 6.1, 6.2**, **Lecture Problem**)

Consider the following binary hypothesis testing scenario. (Note that all required integrals can be solved by calculating the areas of rectangles and triangles, so we are expecting exact answers.)



The hypothesis probabilities are $\mathbb{P}[H_0] = 2/3$ and $\mathbb{P}[H_1] = 1/3$.

(a) Determine the ML rule.

(b) Determine the MAP rule.

(c) Determine the probability of error under the ML rule.

(d) Determine the probability of error under the MAP rule.

Problem 8.2 (**Video 6.1, 6.2, 6.3**, **Quick Calculations**) For each of the scenarios below, determine the requested quantities.

(a) Under H_0, Y is Gaussian$(-1, 1)$. Under H_1, Y is Gaussian$(+1, 1)$. Let $\mathbb{P}[H_0] = 1/3$ and $\mathbb{P}[H_1] = 2/3$. Determine the ML and MAP decision rules.

(b) Under H_0, Y is Exponential(1). Under H_1, Y is Exponential(2). Let $\mathbb{P}[H_0] = 1/2$ and $\mathbb{P}[H_1] = 1/2$. Determine the likelihood ratio, the ML rule, and the probability of error under the ML rule.

(c) Under H_0, Y is Binomial$(4, 1/2)$. Under H_1, Y is Binomial$(3, 1/2)$. Let $\mathbb{P}[H_0] = 2/3$ and $\mathbb{P}[H_1] = 1/3$. Determine the probability of error under the ML and MAP decision rules.

Problem 8.3 (**Video 6.1, 6.2, 6.3**)

This problem is meant to walk through some of the steps carried out algorithmically in Lab 8. (It will likely be easier to complete this problem once you have finished at least the first half of the lab.) Below are tables of training and testing data. You will use the training data to determine the Gaussian ML rule and then use the testing data to evaluation its performance.

Table 1: Training Data

Y	label
3.2	1
0.8	1
0.1	0
-2.1	0

Table 2: Testing Data

Y	label
2.0	1
0.7	1
0.1	1
1.1	0
-2.9	0
-1.6	0

(a) Use the training data to estimate the mean μ_0 under label 0 and the mean μ_1 under label 1. This can be done by simply averaging the training with that label.

(b) Assuming the variances under label 0 and 1 are equal to σ^2, determine the ML rule $D_{\text{ML}}(y)$. Specifically, assume that the data Y with label 0 is generated according to Gaussian(μ_0, σ^2) and the data Y with label 1 is generated according to Gaussian(μ_1, σ^2).

(c) Sketch the conditional PDFs Gaussian(μ_0, σ^2) and Gaussian(μ_1, σ^2) on the same plot along with the decision boundary from the ML rule. (You can assume $\sigma^2 = 1$.)

(d) Add the testing data to your sketch just as in Lab 8. Use circles for testing data with label 0 and squares for testing data with label 1.

(e) Draw a star inside each testing data on the plot that will be misclassified by the ML rule.

(f) Estimate the probability of error as the fraction of misclassified points.

Problem 8.4 (**Video 7.1, 7.2**, **Lecture Problem**)

Consider the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{4}{\pi} & x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note this is a uniform distribution over a quarter disk of radius 1.

(a) Determine the MMSE estimator $\hat{x}_{\text{MMSE}}(y)$ of X given $Y = y$.

(b) Determine the Mean Square Error of the MMSE estimator $\mathbb{E}\big[\big(X - \hat{x}_{\text{MMSE}}(Y)\big)^2\big]$.

(c) Let X, Y be joint Gaussian random variables, with zero mean, and $\mathsf{Var}[X] = \mathsf{Var}[Y] = 2$, $\mathsf{Cov}[X,Y] = 1$. Determine the MMSE estimator $\hat{x}_{\text{MMSE}}(y)$ of X given $Y = y$.

(d) Determine the Mean Square Error of the MMSE estimator in the previous part, $\mathbb{E}\big[\big(X - \hat{x}_{\text{MMSE}}(Y)\big)^2\big]$.