EK381 Probability, Statistics, and Data Science for Engineers
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Boston University
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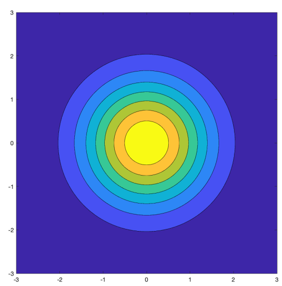
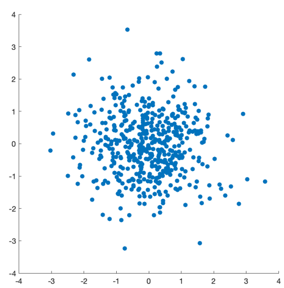
Homework 7: finish by 6/16

Reading: Notes: Chapter 4, 5. **Videos:** 5.1 - 5.6

Problem 7.1 (**Video 5.1, 5.2, 5.3**) In the videos, we introduced the contour plot as a useful way to visualize the joint PDF. The *scatter plot* is a related method for visualizing two-dimensional datasets, which places a marker (e.g., a dot) for each point in the dataset. For example, below are the contour plot and scatter plot (for 500 data points) for a pair of independent Gaussian$(0, 1)$ random variables.





We have provided several scenarios describing pairs of (continuous) random variables via contour plots, scatter plots, and equations. For each of the following parts, list the scenarios that satisfy the specified criteria. For the equations, this will require some quick calculations, but for the contour and scatter plots you should be able to reason visually without calculations.

Solution:

Here are some tricks for working out these comparisons from the plots.
Expected Value: Is more of the mass on the positive side of the x-axis (i.e., right half of the plot)? Then one can say it has $\mathbb{E}[X] > 0$. Is more of the mass on the negative side of y-axis (i.e., bottom half of the plot)? Then one can say it has $\mathbb{E}[Y] < 0$. Does the whole thing look centered at 0 with respect to the x-axis? Then you might have $\mathbb{E}[X]$ close to 0 (which is not noticeably more than 0).
Variance: Is the distribution more stretched out along the x-axis than the y-axis? Then $\mathsf{Var}[X] > \mathsf{Var}[Y]$.
Covariance: Does a line with positive slope fit the data better than a line with negative (or zero) slope? Then you have $\mathsf{Cov}[X, Y] > 0$.
Correlation Coefficient: Does a line explain the data extremely well? Then $|\rho_{X,Y}|$ is close to 1.

(a) $\mathbb{E}[X]$ noticeably more than 0

Solution:

We need a calculation before we list all the scenarios.

For Scenario 7, $\mathbb{E}[X] = \mathbb{E}[U - V] = \mathbb{E}[U] - \mathbb{E}[V] = 1 - 2 = -1$.
Scenarios 1, 4, 6, 9 have $\mathbb{E}[X]$ noticeably more than 0.

(b) $\mathbb{E}[Y]$ noticeably less than 0

Solution:

We need some calculations before we list all the scenarios.
For Scenario 7, $\mathbb{E}[Y] = \mathbb{E}[U + V] = \mathbb{E}[U] + \mathbb{E}[V] = 1 + 2 = 3$.
For Scenario 8, $\mathbb{E}[Y] = \mathbb{E}[X + Z] = \mathbb{E}[X] + \mathbb{E}[Z] = 0 + 0 = 0$.
For Scenario 9, $\mathbb{E}[Y] = \mathbb{E}[-3X + 5] = -3\mathbb{E}[X] + 5 = -3 \cdot 2 = -6$.
Scenarios 2, 3, 9 have $\mathbb{E}[Y]$ noticeably less than 0.

(c) $\mathsf{Var}[X]$ noticeably larger than $\mathsf{Var}[Y]$

Solution:

We need some calculations before we list all the scenarios.
For Scenario 7, $\mathsf{Var}[X] = \mathsf{Var}[U - V] = \mathsf{Var}[U] + \mathsf{Var}[V] - 2\mathsf{Cov}[U, V] = 1 + 1 - 2(-1/2) = 3$
$\mathsf{Var}[Y] = \mathsf{Var}[U + V] = \mathsf{Var}[U] + \mathsf{Var}[V] + 2\mathsf{Cov}[U, V] = 1 + 1 + 2(-1/2) = 1$
For Scenario 8, $\mathsf{Var}[Y] = \mathsf{Var}[X + Z] = \mathsf{Var}[X] + \mathsf{Var}[Z] + 2\mathsf{Cov}[X, Z] = 1 + 3 + 2 \cdot 0 = 4$.
For Scenario 9, $\mathsf{Var}[Y] = \mathsf{Var}[-3X + 5] = (-3)^2\mathsf{Var}[X] = 9 \cdot 1 = 9$.
Scenarios 1, 5, 7 have $\mathsf{Var}[X]$ noticeably larger than $\mathsf{Var}[Y]$.

(d) $\mathsf{Cov}[X, Y]$ noticeably more than 0

Solution:

We need some calculations before we list all the scenarios.
For Scenario 7, $\mathsf{Cov}[X, Y] = \mathsf{Cov}[U - V, U + V]$

$$= 1 \cdot 1 \cdot \mathsf{Var}[U] + (-1) \cdot 1 \cdot \mathsf{Var}[V] + (1 \cdot 1 + (-1) \cdot 1)\mathsf{Cov}[U, V]$$
$$= 1 - 1 + 0 \cdot (-1/2) = 0$$

For Scenario 8, $\mathsf{Cov}[X, Y] = \mathsf{Cov}[X, X + Z]$

$$= 1 \cdot 1 \cdot \mathsf{Var}[X] + 0 \cdot 1 \cdot \mathsf{Var}[Z] + (1 \cdot 1 + 0 \cdot 1)\mathsf{Cov}[X, Z]$$
$$= 1 + 0 + 0 = 1$$

For Scenario 9, $\mathsf{Cov}[X, Y] = \mathsf{Cov}[X, -3X + 5] = 1 \cdot -3\mathsf{Cov}[X, X] = -3\mathsf{Var}[X] = -3$.
Scenarios 1, 2, 5, 6, 8 have $\mathsf{Cov}[X, Y]$ noticeably more than 0.

(e) $|\rho_{X,Y}|$ close to 1

Solution:

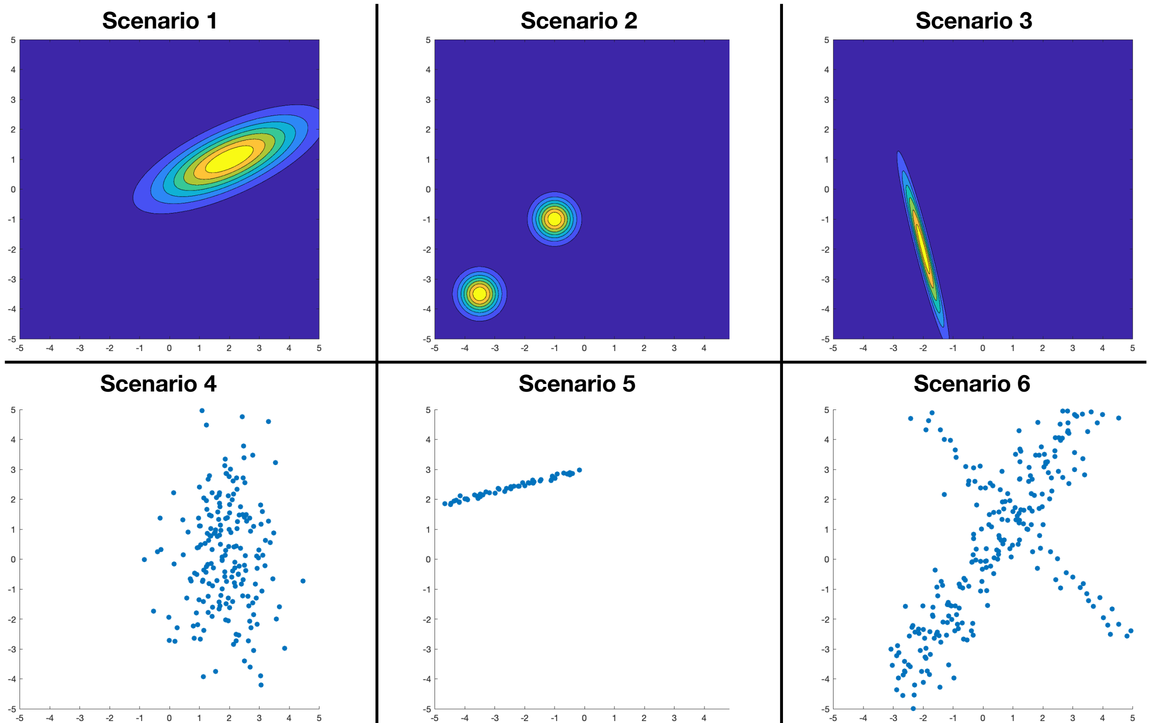
We need some calculations before we list all the scenarios.

For Scenario 7, $\rho_{X,Y} = \dfrac{\mathsf{Cov}[X, Y]}{\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}} = \dfrac{0}{\sqrt{3 \cdot 1}} = 0$

For Scenario 8, $\rho_{X,Y} = \dfrac{\mathsf{Cov}[X, Y]}{\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}} = \dfrac{1}{\sqrt{1 \cdot 4}} = \dfrac{1}{2}$

For Scenario 9, $\rho_{X,Y} = \dfrac{\mathsf{Cov}[X, Y]}{\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}} = \dfrac{-3}{\sqrt{1 \cdot 9}} = -1$

Scenarios 3, 5, 9 have $|\rho_{X,Y}|$ close to 1.



Scenario 7	Scenario 8	Scenario 9
$X = U - V, \quad Y = U + V$ $\mathbb{E}[U] = 1, \ \mathbb{E}[V] = 2,$ $\mathsf{Var}[U] = \mathsf{Var}[V] = 1,$ $\mathsf{Cov}[U, V] = -\frac{1}{2}$	$Y = X + Z$ X is Gaussian$(0, 1)$ Z is Gaussian$(0, 3)$ X, Z independent	$Y = -3X + 5$ $\mathbb{E}[X] = 2, \ \mathsf{Var}[X] = 1$

Problem 7.2 (**Video 5.1 - 5.6**, **Quick Calculations**)

For each of the scenarios below, determine the requested quantities. (You should be able to do this without any long calculations or integration.)

(a) Let X and Y be random variables with $\mathbb{E}[X] = \mathbb{E}[Y] = 1$, $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 5$, and $\rho_{X,Y} = -\frac{1}{4}$. Calculate $\mathsf{Var}[X]$ and $\mathsf{Cov}[X, Y]$.

Solution:

$$\mathsf{Var}[X] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = 5 - 1^2 = 4$$
$$\mathsf{Var}[Y] = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2 = 5 - 1^2 = 4$$
$$\mathsf{Cov}[X, Y] = \rho_{X,Y}\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]} = -\frac{1}{4} \cdot \sqrt{4 \cdot 4} = -1$$

(b) Let X be a random variable with $\mathsf{Var}[X] = 5$ and let $Y = 2X - 3$. Calculate $\mathsf{Cov}[X, Y]$ and $\rho_{X,Y}$.

Solution:

First, note that $\rho_{X,Y} = 1$ since Y is a linear function of X with positive slope. Second, note that $\mathsf{Var}[Y] = 2^2\mathsf{Var}[X] = 4 \cdot 5 = 20$. Finally, we can write $\mathsf{Cov}[X,Y] = \rho_{X,Y}\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]} = 1 \cdot \sqrt{5 \cdot 20} = 1 \cdot 10 = 10$.

(c) Let X and Y be random variables with $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = -1$, $\mathsf{Var}[X] = 4$, $\mathsf{Var}[Y] = 2$, and $\mathsf{Cov}[X,Y] = 1$. Let $W = X + Y$ and $Z = 2X - Y$. Calculate $\mathbb{E}[W + Z]$ and $\mathsf{Cov}[W,Z]$.

Solution:

By the linearity of expectation, $\mathbb{E}[W + Z] = \mathbb{E}[X + Y + 2X - Y] = \mathbb{E}[3X] = 3\mathbb{E}[X] = 3$. Using the formula $\mathsf{Cov}[aX + bY, cX + dY] = ac\,\mathsf{Var}[X] + bd\,\mathsf{Var}[Y] + (ad + bc)\,\mathsf{Cov}[X,Y]$, we have that

$$\begin{aligned}\mathsf{Cov}[W,Z] &= 1 \cdot 2 \cdot \mathsf{Var}[X] + 1 \cdot (-1) \cdot \mathsf{Var}[Y] + (1 \cdot 2 + 1 \cdot (-1))\mathsf{Cov}[X,Y] \\ &= 2 \cdot 4 - 1 \cdot 2 + 1 \cdot 1 = 7\end{aligned}$$

(d) Let X and Y be jointly Gaussian random variables with $\mathbb{E}[X] = 2$, $\mathbb{E}[Y] = -1$, $\mathsf{Var}[X] = 4$, $\mathsf{Var}[Y] = 1$, and $\mathsf{Cov}[X,Y] = -2$. Calculate $\mathbb{P}[2X - Y \leq 1]$ and $\mathbb{P}[X \leq Y]$. (You may leave your answers in terms of the standard normal CDF $\Phi(z)$.)

Solution:

The important thing to keep in mind is that any linear combination of jointly Gaussian random variables is itself a Gaussian random variable. We just need to determine the mean and variance in order to complete the probability calculation. Let $W = 2X - Y$. By the linearity of expectation, $\mathbb{E}[W] = \mathbb{E}[2X - Y] = 2\mathbb{E}[X] - \mathbb{E}[Y] = 2 \cdot 2 - (-1) = 5$. Using the formula $\mathsf{Var}[aX + bY + c] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y] + 2ab\mathsf{Cov}[X,Y]$ for the variance of a linear function,

$$\begin{aligned}\mathsf{Var}[W] = \mathsf{Var}[2X - Y] &= 2^2\mathsf{Var}[X] + (-1)^2\mathsf{Var}[Y] + 2 \cdot 2 \cdot (-1)\mathsf{Cov}[X,Y] \\ &= 4 \cdot 4 + 1 \cdot 1 - 4 \cdot (-2) = 25\ .\end{aligned}$$

Now, we can calculate

$$\mathbb{P}[2X - Y \leq 1] = \mathbb{P}[W \leq 1] = F_W(1) = \Phi\left(\frac{1 - \mathbb{E}[W]}{\sqrt{\mathsf{Var}[W]}}\right) = \Phi\left(\frac{1-5}{\sqrt{25}}\right) = \Phi\left(-\frac{4}{5}\right)\ .$$

Note that $\mathbb{P}[X \leq Y] = \mathbb{P}[X - Y \leq 0]$ so we can follow a similar process for $V = X - Y$.

$$\begin{aligned}\mathbb{E}[V] &= \mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] = 2 - (-1) = 3 \\ \mathsf{Var}[V] = \mathsf{Var}[X - Y] &= 1^2\mathsf{Var}[X] + (-1)^2\mathsf{Var}[Y] + 2 \cdot 1 \cdot (-1)\mathsf{Cov}[X,Y] \\ &= 1 \cdot 4 + 1 \cdot 1 - 2 \cdot (-2) = 9\ .\end{aligned}$$

Now, we can calculate

$$\mathbb{P}[X \le Y] = \mathbb{P}[V \le 0] = F_V(0) = \Phi\left(\frac{0 - \mathbb{E}[V]}{\sqrt{\mathsf{Var}[V]}}\right) = \Phi\left(\frac{0-3}{\sqrt{9}}\right) = \Phi(-1)\ .$$

(e) Let X and Y be jointly Gaussian with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathsf{Var}[X] = \mathsf{Var}[Y] = 2$, and $\mathsf{Cov}[X,Y] = 1$. Calculate $\mathbb{E}[X|Y = 2]$ and $\mathbb{P}[X < 0|Y = 2]$. (You may leave your second answer in terms of the $\Phi(z)$ function.)

Solution:
X given that $Y = 2$ is a Gaussian($\mathbb{E}[X|Y = 2]$, $\mathsf{Var}[X|Y = 2]$).

$$\mathbb{E}[X|Y = 2] = \mathbb{E}[X] + \frac{\mathsf{Cov}[X,Y]}{\mathsf{Var}[Y]}(Y - \mathbb{E}[Y]) = 0 + \frac{1}{2}(2-0) = 1$$
$$\mathsf{Var}[X|Y = 2] = \mathsf{Var}[X] - \frac{(\mathsf{Cov}[X,Y])^2}{\mathsf{Var}[Y]} = 2 - \frac{1^2}{2} = \frac{3}{2}$$
$$\mathbb{P}[X < 0|Y = 2] = \Phi\left(\frac{0 - \mathbb{E}[X|Y = 2]}{\sqrt{\mathsf{Var}[X|Y = 2}}\right) = \Phi\left(\frac{0-1}{\sqrt{\frac{3}{2}}}\right) = \Phi\left(-\sqrt{\frac{2}{3}}\right)$$

(f) Let X_1 and X_2 be jointly Gaussian with $\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$, $\mathsf{Var}[X_1] = \mathsf{Var}[X_2] = 1$, and $\mathsf{Cov}[X_1, X_2] = \frac{1}{3}$. Define $Y_1 = 2X_1 + 1$ and $Y_2 = X_1 - X_2 - 1$. Define the random vector $\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$. Calculate the mean vector $\underline{\mu}_{\underline{Y}}$ and the covariance matrix $\mathbf{\Sigma}_{\underline{Y}}$.

Solution:

For the mean vector, we use linearity of expectation,

$$\underline{\mu}_{\underline{Y}} = \begin{bmatrix} \mathbb{E}[Y_1] \\ \mathbb{E}[Y_2] \end{bmatrix} = \begin{bmatrix} \mathbb{E}[2X_1 + 1] \\ \mathbb{E}[X_1 - X_2] \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 + 1 \\ 0 - 0 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

First, note that $\mathbf{\Sigma}_{\underline{X}} = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}$ and that $\underline{Y} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \underline{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Using the covariance of a linear transform, we have that $\mathbf{\Sigma}_{\underline{Y}} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 4 & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} \end{bmatrix}$

Problem 7.3 (**Video 5.1, 5.2, 5.3, 5.4**, **Lecture Problem**)

A robot is tasked with moving 4 meters in both the x-direction and y-direction, starting from the origin. However, due to wheel slippage and uneven terrain, its actual movement deviates from the intended path. These deviations are modeled by a bivariate Gaussian distribution, where the errors in both directions are positively correlated. Specifically, (X, Y) is jointly Gaussian with means $\mathbb{E}[X] = \mathbb{E}[Y] = 4$, variances $\mathsf{Var}[X] = \mathsf{Var}[Y] = 0.25$, and $\mathsf{Cov}[X,Y] = 0.1$.

(a) Compute the expected value $\mathbb{E}[W]$ and variance $\mathsf{Var}[W]$ of the robot's movement in the direction $W = \frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y$.

Solution:

Using the linearity of expectation, we find that

$$\mathbb{E}[W] = \mathbb{E}\left[\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y\right] = \frac{1}{\sqrt{2}}\mathbb{E}[X] + \frac{1}{\sqrt{2}}\mathbb{E}[Y] = \frac{1}{\sqrt{2}}\cdot 4 + \frac{1}{\sqrt{2}}\cdot 4 = \frac{8}{\sqrt{2}} = 4\sqrt{2} \approx 5.567 \,.$$

The variance follows from the formula $\mathsf{Var}[aX + bY + c] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y] + 2ab\mathsf{Cov}[X, Y]$ for the variance of a linear combination:

$$\begin{aligned}\mathsf{Var}[W] = \mathsf{Var}\left[\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y\right] &= \frac{1}{2}\mathsf{Var}[X] + \frac{1}{2}\mathsf{Var}[Y] + 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\mathsf{Cov}[X, Y] \\ &= \frac{1}{2}\cdot 0.25 + \frac{1}{2}\cdot 0.25 + 0.1 = 0.35\end{aligned}$$

(b) Calculate the probability that the robot's movement along this direction exceeds 6 meters.

Solution:

Since a linear combination of jointly Gaussian random variables is itself Gaussian, we know that W is Gaussian$(4\sqrt{2}, 0.35)$, since we calculated the mean and variance in part (a). The problem is asking for $\mathbb{P}[W > 6]$, which we can calculate using the $\Phi(z)$ function.

$$\mathbb{P}[W > 6] = 1 - \mathbb{P}[W \leq 6] = 1 - \Phi\left(\frac{6 - 4\sqrt{2}}{\sqrt{0.35}}\right) = 1 - \Phi(0.580) \approx 1 - 0.719 = 0.281$$

(c) Suppose that, after the robot has completed its movement, we determine that its exact position along the y-axis is 4.5 meters. Find the conditional distribution of the robot's position X given that Y = 4.5. Don't forget the parameters.

Solution:

The first step is to recognize that, for jointly Gaussian X and Y, the conditional PDF for X given $Y = y$ is Gaussian with mean taken from the jointly Gaussian conditional expectation formula $\mathbb{E}[X|Y = y] = \mathbb{E}[X] + \frac{Cov[X,Y]}{\mathsf{Var}[Y]}(y - \mathbb{E}[Y])$ and variance taken from the jointly Gaussian conditional variance formula $\mathsf{Var}[X|Y = y] = \mathsf{Var}[X] - \frac{(\mathsf{Cov}[X,Y])^2}{\mathsf{Var}[Y]}$. In our setting, we plug in the parameters and set $y = 4.5$,

$$\begin{aligned}\mathbb{E}[X|Y = 4.5] &= 4 + \frac{0.1}{0.25}(4.5 - 4) = 4.2 \\ \mathsf{Var}[X|Y = 4.5] &= 0.25 - \frac{(0.1)^2}{0.25} = 0.21 \,,\end{aligned}$$

and we find that X given $Y = 4.5$ is Gaussian$(4.2, 0.21)$.

(d) More generally, for a given y position, compute the conditional expected value of its x position, $E[X|Y = y]$.

Solution:

We can again use the jointly Gaussian conditional expectation formula $\mathbb{E}[X|Y=y] = \mathbb{E}[X] + \frac{Cov[X,Y]}{\mathsf{Var}[Y]}(y - \mathbb{E}[Y])$ to get

$$\mathbb{E}[X|Y=y] = 4 + \frac{0.1}{0.25}(y-4) = 0.4\,y + 2.4\ .$$

Problem 7.4 (**Video 5.5, 5.6**, **Lecture Problem**)

Jointly Gaussian random variables play an important role in probability theory, due partly to the fact that linear combinations of Gaussians are themselves Gaussian. This allows us to answer complex questions by only calculating means and variances. Here, we will explore an application of this phenomenon to the expected value. Let $X_1, \ldots, X_n$ be independent Gaussian random variables, with expected values $\mathbb{E}[X_i] = \mu$ and variances $\mathsf{Var}[X_i] = \sigma^2$ for $i = 1, \ldots, n$ and let $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ be their average. Intuitively, the random variable Y should get "closer" to μ as the number of samples n increases. Below, we will try to make this intuition precise.

(a) Define $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$. Determine its mean vector $\mathbb{E}[\underline{X}]$ and covariance matrix $\mathbf{\Sigma}_{\underline{X}}$.

Solution:

$$\mathbb{E}[\underline{X}] = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} \qquad\qquad \mathbf{\Sigma}_{\underline{X}} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

The diagonal entries of the covariance matrix are given directly by the variances, and we know the off-diagonal entries are zero since $X_1, \ldots, X_n$ are independent.

(b) Express $Y = \mathbf{A}\underline{X}$ for some matrix $\mathbf{A}$. (Note that row vectors and column vectors are special cases of matrices.)

Solution:

With $\mathbf{A} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$, we have $Y = \mathbf{A}\underline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(c) Determine $\mathbb{E}[Y]$.

Solution:

Using the linearity of expectation,

$$\mathbb{E}[Y] = \mathbb{E}[\mathbf{A}\underline{X}] = \mathbf{A}\mathbb{E}[\underline{X}] = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} = n \cdot \frac{1}{n} \cdot \mu = \mu \ .$$

Therefore, the expected value of the average is μ, matching our intuition.

(d) Determine $\mathsf{Var}[Y]$.

Solution:

Note that we can think about Y as a 1×1 matrix with covariance matrix $\mathbf{\Sigma}_Y = \mathsf{Var}[Y]$. Now, using the formula for the covariance of a linear transformation, we have that

$$\begin{aligned} \mathsf{Var}[Y] = \mathbf{A}\mathbf{\Sigma}_{\underline{X}}\mathbf{A}^{\mathsf{T}} &= \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{n} \\ \frac{\sigma^2}{n} \\ \vdots \\ \frac{\sigma^2}{n} \end{bmatrix} = n \cdot \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n} \end{aligned}$$

Notice that the variance is decreasing as a function of n, meaning Y varies less around μ as we include more samples in the average, again matching our intuition.

(e) Calculate the probability that Y is more than $\delta > 0$ away from its mean, $\mathbb{P}\big[|Y - \mathbb{E}[Y]| > \delta\big]$. Express your answer in terms of the standard normal complementary CDF $Q(z) = 1 - \Phi(z)$.

Solution:

Since Y is a linear combination of jointly Gaussian random variables, it is itself Gaussian. Therefore,

$$\begin{aligned} \mathbb{P}\big[|Y - \mathbb{E}[Y]| > \delta\big] &= \mathbb{P}\big[\{Y > \mathbb{E}[Y] + \delta\} \cup \{Y < \mathbb{E}[Y] - \delta\}\big] \\ &= \mathbb{P}[Y > \mathbb{E}[Y] + \delta] + \mathbb{P}[Y < \mathbb{E}[Y] - \delta] \\ &= 1 - F_Y(\mu + \delta) + F_Y(\mu - \delta) \\ &= 1 - \Phi\left(\frac{\mu + \delta - \mu}{\sqrt{\sigma^2/n}}\right) + \Phi\left(\frac{\mu - \delta - \mu}{\sqrt{\sigma^2/n}}\right) \\ &= 1 - \Phi\left(\frac{\delta\sqrt{n}}{\sigma}\right) + \Phi\left(-\frac{\delta\sqrt{n}}{\sigma}\right) \\ &= 2Q\left(\frac{\delta\sqrt{n}}{\sigma}\right) \end{aligned}$$

Note that, since $\Phi(z) \to 0$ as $z \to \infty$, we have that the probability that Y is more than

δ away from its mean goes to 0 as the number of samples n increases, for any choice of $\delta > 0$.

(f) Using the fact that $Q(3.29) = \frac{1}{2000}$, calculate how many samples n are needed, as a function of the variance σ^2 to guarantee that $\mathbb{P}\big[\big|Y - \mathbb{E}[Y]\big| > \frac{1}{10}\big]$ is $\frac{1}{1000}$ or smaller.

Solution:

$$\mathbb{P}\big[\big|Y - \mathbb{E}[Y]\big| > \delta\big] = 2Q\left(\frac{\delta\sqrt{n}}{\sigma}\right) = \frac{1}{1000}$$
$$\implies Q\left(\frac{\delta\sqrt{n}}{\sigma}\right) = \frac{1}{2000}$$

Therefore, we just need to set $\frac{\delta\sqrt{n}}{\sigma} = 3.29$ and $\delta = \frac{1}{10}$. It follows that

$$\frac{\frac{1}{10}\sqrt{n}}{\sigma} = 3.29 \implies \sqrt{n} = 32.9 \cdot \sigma \implies n = 1082.41 \cdot \sigma^2$$