
Homework 6: finish by 6/12.

Reading: Notes: Chapter 4, 5. **Videos:** 4.7, 5.1 - 5.2

Problem 6.1 (**Video 4.1 - 4.7**, **Lecture Problem**)

Let X be Uniform$[1,2]$. Let Y given $X = x$ be Exponential(x). That is, let

$$f_{Y|X}(y|x) = \begin{cases} xe^{-xy} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

(a) Find the expected value of Y. (It is OK to leave your answer as an integral.)

> **Solution:**
>
> $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \displaystyle\int_1^2 xe^{-xy}\,dx & y \geq 0 \\ 0 & y < 0 \end{cases}$$
>
> $$\begin{aligned} \mathbb{E}[Y] &= \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^\infty y\left(\int_1^2 xe^{-xy}\,dx\right)dy = \int_0^\infty \int_1^2 xye^{-xy}\,dx\,dy \\ &= \int_1^2 \int_0^\infty xye^{-xy}\,dy\,dx = \int_1^2 \left(\int_0^\infty yxe^{-xy}\,dy\right)dx = \int_1^2 \frac{1}{x}\,dx = \ln(2) \end{aligned}$$

(b) Find the conditional expected value $\mathbb{E}[Y|X=x]$ of Y given $X = x$. (This should be in closed form.)

> **Solution:**
>
> Recall that an Exponential(λ) random variable has expected value $\frac{1}{\lambda}$.
> Therefore, $\mathbb{E}[Y|X=x] = \frac{1}{x}$.

(c) Using $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$ find the expected value of Y. (This should be in closed form.)

> **Solution:**
>
> $$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x]\, f_X(x)\,dx = \int_1^2 \frac{1}{x}\,dx = \ln(x)\Big|_1^2 = \ln(2)$$

(d) Solve for $\mathbb{E}[XY]$. (It is OK to leave your answer as an integral.)

Solution:

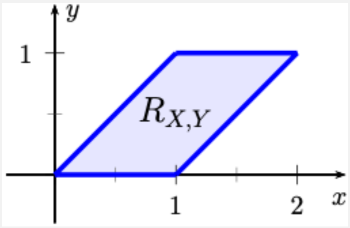
$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy = \int_0^{\infty}\int_1^2 x^2 y e^{-xy}\, dx\, dy$$

Problem 6.2 (**Video 4.6, 4.7, 5.1, 5.2**, **Lecture Problem**)

Consider the joint PDF $f_{X,Y}(x,y) = \begin{cases} x & y \le x \le y+1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(a) Draw a sketch of the range $R_{X,Y}$.

Solution:



(b) What is the probability that $2Y$ is greater than X?

Solution:

$$\mathbb{P}[2Y > X] = \int_{-\infty}^{\infty}\int_{-\infty}^{2y} f_{X,Y}\, dx\, dx = \int_0^1\int_y^{2y} x\, dx\, dy$$
$$= \int_0^1 \left(\frac{1}{2}x^2\right)\Big|_y^{2y} dy = \int_0^1 \frac{3}{2}y^2\, dy = \frac{1}{2}y^3\Big|_0^1 = \frac{1}{2}$$

(c) Are X and Y independent? Explain.

Solution:

No, the range is not a rectangle, so the range cannot factor and thus the joint PDF does not factor either.

(d) Compute $E[X^2]$.

Solution:

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\, dx\, dy = \int_0^1\int_y^{y+1} x^3\, dx\, dy$$
$$= \int_0^1 \left(\frac{1}{4}x^4\right)\Big|_y^{y+1} dy = \int_0^1 \left(\frac{(y+1)^4}{4} - \frac{y^4}{4}\right) dy = \int_0^1 \left(\frac{4y^3+6y^2+4y+1}{4}\right) dy$$

$$= \left(\frac{y^4 + 2y^3 + 2y + y}{4} \right) \Bigg|_0^1 = \frac{3}{2}$$

(e) Compute $\mathsf{Cov}[X, Y]$.

Solution:

We will use the formula $\mathsf{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. (For full credit, it suffices to give this formula accompanied by the integrals in blue below.)

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{X,Y}\, dx\, dx = \int_0^1 \int_y^{y+1} x^2 y\, dx\, dy \\
&= \int_0^1 \left(\frac{1}{3} x^3 \right) \Bigg|_y^{y+1} y\, dy = \int_0^1 \left(\frac{(y+1)^3}{3} - \frac{y^3}{3} \right) y\, dy = \int_0^1 \left(y^3 + y^2 + \frac{y}{3} \right) dy \\
&= \left(\frac{y^4}{4} + \frac{y^3}{3} + \frac{y^2}{6} \right) \Bigg|_0^1 = \frac{3}{4} \\
\mathbb{E}[X] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X,Y}\, dx\, dx = \int_0^1 \int_y^{y+1} x^2\, dx\, dy \\
&= \int_0^1 \left(\frac{1}{3} x^3 \right) \Bigg|_y^{y+1} dy = \int_0^1 \left(\frac{(y+1)^3}{3} - \frac{y^3}{3} \right) dy = \int_0^1 \left(\frac{3y^2 + 3y + 1}{3} \right) dy \\
&= \left(\frac{y^3}{3} + \frac{y^2}{2} + \frac{y}{3} \right) \Bigg|_0^1 = \frac{7}{6} \\
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{X,Y}\, dx\, dx = \int_0^1 \int_y^{y+1} xy\, dx\, dy \\
&= \int_0^1 \left(\frac{1}{2} x^2 \right) \Bigg|_y^{y+1} y\, dy = \int_0^1 \left(\frac{(y+1)^2}{2} - \frac{y^2}{2} \right) y\, dy = \int_0^1 \left(y^2 + \frac{y}{2} \right) dy \\
&= \left(\frac{y^3}{3} + \frac{y^2}{4} \right) \Bigg|_0^1 = \frac{7}{12}
\end{aligned}$$

Putting this together, we have $\mathsf{Cov}[X, Y] = \frac{3}{4} - \frac{7}{6} \cdot \frac{7}{12} = \frac{5}{72}$.

(f) Compute $\rho_{X,Y}$.

Solution:

We can use the formula $\rho_{X,Y} = \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}}$. To obtain the variances, we will use $\mathsf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ and $\mathsf{Var}[Y] = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2$. The only integral we are missing is

$$\begin{aligned}
\mathbb{E}[Y^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2\, f_{X,Y}(x, y)\, dx\, dy = \int_0^1 \int_y^{y+1} xy^2\, dx\, dy \\
&= \int_0^1 \left(\frac{1}{2} x^2 \right) \Bigg|_y^{y+1} y^2\, dy = \int_0^1 \left(\frac{(y+1)^2}{2} - \frac{y^2}{2} \right) y^2\, dy = \int_0^1 \left(y^3 + \frac{y^2}{2} \right) dy
\end{aligned}$$

$$= \left(\frac{y^4}{4} + \frac{y^3}{6}\right)\Bigg|_0^1 = \frac{5}{12}$$

Putting everything together, we have

$$\begin{aligned}
\mathsf{Var}[X] &= \frac{3}{2} - \left(\frac{7}{6}\right)^2 = \frac{5}{36} \\
\mathsf{Var}[Y] &= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144} \\
\rho_{X,Y} &= \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\,\mathsf{Var}[Y]}} = \frac{\frac{5}{72}}{\sqrt{\frac{5}{36}\cdot\frac{11}{144}}} = \sqrt{\frac{5}{11}} \approx 0.674
\end{aligned}$$

(g) Compute $f_{X|Y}(x|y)$.

Solution:

First, we will need the marginal PDF of Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \displaystyle\int_y^{y+1} x\,dx & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can use the conditional PDF formula:

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{x}{y+\frac{1}{2}} & y \le x \le y+1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(h) Compute $\mathbb{E}[2X+1|Y]$.

Solution:

By the linearity of expectation, $\mathbb{E}[2X+1|Y] = 2\mathbb{E}[X|Y]+1$. The conditional expected value of X given $Y = y$ is

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x\,f_{X|Y}(x|y)\,dx = \int_y^{y+1} \frac{x^2}{y+\frac{1}{2}}\,dx = \frac{(y+1)^3 - y^3}{3(y+\frac{1}{2})} = \frac{2y^2+2y+\frac{2}{3}}{2y+1}.$$

Therefore, we have that

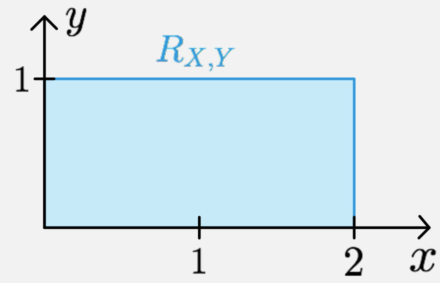
$$\mathbb{E}[2X+1|Y] = 2\frac{2Y^2+2Y+\frac{2}{3}}{2Y+1} + 1 = \frac{4Y^2+4Y+\frac{4}{3}+2Y+1}{2Y+1} = \frac{4Y^2+6Y+\frac{7}{3}}{2Y+1}.$$

Problem 6.3 (**Video 4.6, 4.7, 5.1, 5.2**)

Consider the following joint PDF $f_{X,Y}(x,y) = \begin{cases} \dfrac{x}{4} + \dfrac{y}{2} & 0 \le x \le 2,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Draw a sketch of the range $R_{X,Y}$.

Solution:



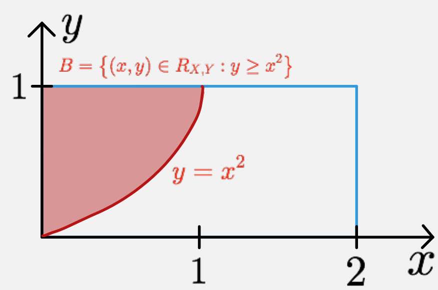
(b) Are X and Y independent?

Solution:

No, they are not independent. Even though the range is a rectangle, the joint PDF cannot be written as a product of the marginal PDFs (as we will see below).

(c) What is the probability that Y is greater than X^2?

Solution:



$$\begin{aligned}
\mathbb{P}[Y > X^2] &= \int_{-\infty}^{\infty} \int_{x^2}^{\infty} f_{X,Y}(x, y)\, dy\, dx = \int_0^1 \int_{x^2}^1 \left(\frac{x}{4} + \frac{y}{2}\right) dy\, dx \\
&= \int_0^1 \left(\frac{xy}{4} + \frac{y^2}{4}\right)\Big|_{x^2}^1 dx = \int_0^1 \left(\frac{x}{4} + \frac{1}{4} - \frac{x^3}{4} - \frac{x^4}{4}\right) dx \\
&= \left(\frac{x^2}{8} + \frac{x}{4} - \frac{x^4}{16} - \frac{x^5}{20}\right)\Big|_0^1 = \frac{1}{8} + \frac{1}{4} - \frac{1}{16} - \frac{1}{20} = \frac{21}{80}
\end{aligned}$$

(d) Determine the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^1 \left(\frac{x}{4} + \frac{y}{2}\right) dy & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left(\frac{xy}{4} + \frac{y^2}{4}\right)\Big|_0^1 & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{x}{4} + \frac{1}{4} & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^2 \left(\frac{x}{4} + \frac{y}{2}\right) dx & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left(\frac{x^2}{8} + \frac{xy}{2}\right)\Big|_0^2 & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(e) Calculate $\mathsf{Cov}[X, Y]$.

Solution:

We will use the formula $\mathsf{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^2 \left(\frac{x^2}{4} + \frac{x}{4}\right) dx = \left(\frac{x^3}{12} + \frac{x^2}{8}\right)\Big|_0^2 = \frac{8}{12} + \frac{4}{8} = \frac{7}{6}$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dx = \int_0^1 \left(y^2 + \frac{y}{2}\right) dy = \left(\frac{y^3}{3} + \frac{y^2}{4}\right)\Big|_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\mathbb{E}[XY] = \int_0^2 \int_0^1 xy\, f_{X,Y}(x,y)\,dy\,dx = \int_0^2 \int_0^1 \left(\frac{x^2 y}{4} + \frac{xy^2}{2}\right) dy\,dx$$

$$= \int_0^2 \left(\frac{x^2 y^2}{8} + \frac{xy^3}{6}\right)\Big|_0^1 dx = \int_0^2 \left(\frac{x^2}{8} + \frac{x}{6}\right) dx = \left(\frac{x^3}{24} + \frac{x^2}{12}\right)\Big|_0^2 = \frac{8}{24} + \frac{4}{12} = \frac{2}{3}$$

$$\mathsf{Cov}[X, Y] = \frac{2}{3} - \frac{7}{6} \cdot \frac{7}{12} = -\frac{1}{72}$$

(f) Calculate $\rho_{X,Y}$.

Solution:

To compute the correlation coefficient, $\rho_{X,Y} = \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]}}$, we first need to compute the variances. Using the formulas $\mathsf{Var}[X] = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$ and $\mathsf{Var}[Y] = \mathbb{E}[Y^2] - \left(\mathbb{E}[Y]\right)^2$, we see that we just need to calculate the second moments.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^2 \left(\frac{x^3}{4} + \frac{x^2}{4}\right) dx = \left(\frac{x^4}{16} + \frac{x^3}{12}\right)\Big|_0^2 = \frac{16}{16} + \frac{8}{12} = \frac{5}{3}$$

$$\mathbb{E}[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dx = \int_0^1 \left(y^3 + \frac{y^2}{2}\right) dy = \left(\frac{y^4}{4} + \frac{y^3}{6}\right)\Big|_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Putting things together, we get

$$\mathsf{Var}[X] = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{11}{36}$$
$$\mathsf{Var}[Y] = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$
$$\rho_{X,Y} = \frac{-\frac{1}{72}}{\sqrt{\frac{11}{36}\cdot\frac{11}{144}}} = -\frac{6\cdot 12}{11\cdot 72} = -\frac{1}{11}$$

(g) Determine the conditional PDF $f_{X|Y}(x|y)$.

Solution:

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) > 0 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \dfrac{\frac{x}{4}+\frac{y}{2}}{y+\frac{1}{2}} & 0 \le x \le 2,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(h) Calculate the conditional expected value $\mathbb{E}[X|Y=y]$.

Solution:

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$
$$= \int_0^2 x\frac{\frac{x}{4}+\frac{y}{2}}{y+\frac{1}{2}}\,dx = \frac{1}{y+\frac{1}{2}}\int_0^2\left(\frac{x^2}{4}+\frac{xy}{2}\right)dx = \frac{1}{y+\frac{1}{2}}\left(\frac{x^3}{12}+\frac{x^2y}{4}\right)\Big|_0^2 = \frac{y+\frac{2}{3}}{y+\frac{1}{2}}$$

(i) Using the iterated expectation formula $\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[X]$ and your answer from part (h), determine $\mathbb{E}[X]$. Compare with the answer you found from the marginals as part of your calculations for part (e).

Solution:

From the iterated expectation formula,

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big] = \int_{-\infty}^{\infty}\mathbb{E}[X|Y=y]f_Y(y)dy$$
$$= \int_0^1 \frac{y+\frac{2}{3}}{y+\frac{1}{2}}\left(y+\frac{1}{2}\right)dy = \int_0^1\left(y+\frac{2}{3}\right)dy = \left(\frac{1}{2}y^2+\frac{2y}{3}\right)\Big|_0^1 = \frac{1}{2}+\frac{2}{3} = \frac{7}{6},$$

which agrees with our calculations from part (e).

Problem 6.4 (**Video 4.5, 4.7**, **Quick Calculations**)

Parts (a) - (d): For each of the scenarios below, determine if X and Y are independent, and give a justification. (You should be able to do this without any long calculations or integration.)

(a) $f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Solution:

Not independent since the range is not a rectangle.

(b) Y given X is Geometric($\frac{x+1}{3}$) and X is Bernoulli(1/2).

Solution:

Not independent since $P_{Y|X}(y|x)$ depends on x and thus cannot be equal to the marginal $P_Y(y)$.

(c) $f_{X,Y}(x,y) = \begin{cases} \frac{9}{38}x^2y^2 & -1 \leq x \leq 1, 2 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$

Solution:

Yes, since the range is a rectangle *and* the function will clearly factor into components of the form c_1x^2 and c_2y^2 for some constants c_1 and c_2. Therefore, we can conclude that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

(d)

$P_{XY}(x,y)$		$y = 0$	$y = 1$
x	0	1/15	4/15
	1	2/15	8/15

Solution:

Yes, we can solve for the marginals $P_X(x) = \begin{cases} \frac{1}{3} & x = 0 \\ \frac{2}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$ and $P_Y(y) = \begin{cases} \frac{1}{5} & y = 0 \\ \frac{4}{5} & y = 1 \\ 0 & \text{otherwise} \end{cases}$ and then verify that $P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$.

(e) $f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ \frac{1}{2} & -1 \leq x \leq 0, -1 \leq y \leq 0 \\ 0 & \text{otherwise} \end{cases}$

Solution:

No, the range is not a rectangle, and thus the joint PDF will not factor into the product of the marginal PDFs.

Parts (e) - (f): For each of the scenarios below, determine the requested quantities. (You should be able to do this without any long calculations or integration.)

(e) Let X be Poisson(3) and Y given $X = x$ be Binomial($x, \frac{1}{3}$). Calculate $\mathbb{E}[Y|X = x]$ and $\mathbb{E}[Y]$.

Solution:

For a Binomial(n, p) random variable, the expected value is np. Therefore, $\mathbb{E}[Y|X = x] = \dfrac{x}{3}$. Using the iterated expectation formula,

$$\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big] = \mathbb{E}\left[\frac{X}{3}\right] = \frac{1}{3}\mathbb{E}[X] = \frac{1}{3} \cdot 3 = 1 \ .$$

(f) Let X be Uniform$(\frac{1}{2}, \frac{5}{2})$. Let Y given $X = x$ be Exponential$(\frac{1}{2x})$. Calculate $\mathbb{E}[Y|X = x]$ and $\mathbb{E}[Y]$.

Solution:

First, recall that for an Exponential(λ) random variable, the mean is $\frac{1}{\lambda}$ and the variance is $\frac{1}{\lambda^2}$. Therefore,

$$\mathbb{E}[Y|X = x] = \frac{1}{\frac{1}{2x}} = 2x \ .$$

Using the iterated expectation formula,

$$\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big] = \mathbb{E}[2X] = 2 \cdot \frac{\frac{5}{2} + \frac{1}{2}}{2} = 2 \cdot 3 = 6 \ .$$