EK381 Probability, Statistics, and Data Science for Engineers — Boston University
Prof Carruthers — Summer 2025

Homework 6: finish by 6/12.

Reading: Notes: Chapter 4, 5. **Videos:** 4.7, 5.1 - 5.2

Problem 6.1 (**Video 4.1 - 4.7**, **Lecture Problem**)

Let X be Uniform$[1, 2]$. Let Y given $X = x$ be Exponential(x). That is, let

$$f_{Y|X}(y|x) = \begin{cases} xe^{-xy} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

(a) Find the expected value of Y. (It is OK to leave your answer as an integral.)

(b) Find the conditional expected value $\mathbb{E}[Y|X = x]$ of Y given $X = x$. (This should be in closed form.)

(c) Using $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$ find the expected value of Y. (This should be in closed form.)

(d) Solve for $\mathbb{E}[XY]$. (It is OK to leave your answer as an integral.)

Problem 6.2 (**Video 4.6, 4.7, 5.1, 5.2**, **Lecture Problem**)

Consider the joint PDF $f_{X,Y}(x, y) = \begin{cases} x & y \leq x \leq y + 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

(a) Draw a sketch of the range $R_{X,Y}$.

(b) What is the probability that $2Y$ is greater than X?

(c) Are X and Y independent? Explain.

(d) Compute $E[X^2]$.

(e) Compute $\mathsf{Cov}[X, Y]$.

(f) Compute $\rho_{X,Y}$.

(g) Compute $f_{X|Y}(x|y)$.

(h) Compute $\mathbb{E}[2X + 1|Y]$.

Problem 6.3 (**Video 4.6, 4.7, 5.1, 5.2**)

Consider the following joint PDF $f_{X,Y}(x, y) = \begin{cases} \dfrac{x}{4} + \dfrac{y}{2} & 0 \leq x \leq 2,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Draw a sketch of the range $R_{X,Y}$.

(b) Are X and Y independent?

(c) What is the probability that Y is greater than X^2?

(d) Determine the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(e) Calculate $\mathsf{Cov}[X, Y]$.

(f) Calculate $\rho_{X,Y}$.

(g) Determine the conditional PDF $f_{X|Y}(x|y)$.

(h) Calculate the conditional expected value $\mathbb{E}[X|Y = y]$.

(i) Using the iterated expectation formula $\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[X]$ and your answer from part (h), determine $\mathbb{E}[X]$. Compare with the answer you found from the marginals as part of your calculations for part (e).

Problem 6.4 (**Video 4.5, 4.7**, **Quick Calculations**)

Parts (a) - (d): For each of the scenarios below, determine if X and Y are independent, and give a justification. (You should be able to do this without any long calculations or integration.)

(a) $f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

(b) Y given X is Geometric($\frac{x+1}{3}$) and X is Bernoulli(1/2).

(c) $f_{X,Y}(x, y) = \begin{cases} \frac{9}{38}x^2y^2 & -1 \leq x \leq 1, 2 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$

(d)

$P_{XY}(x, y)$		$y = 0$	$y = 1$
x	0	1/15	4/15
	1	2/15	8/15

Parts (e) - (f): For each of the scenarios below, determine the requested quantities. (You should be able to do this without any long calculations or integration.)

(e) Let X be Poisson(3) and Y given $X = x$ be Binomial($x, \frac{1}{3}$). Calculate $\mathbb{E}[Y|X = x]$ and $\mathbb{E}[Y]$.

(f) Let X be Uniform($\frac{1}{2}, \frac{5}{2}$). Let Y given $X = x$ be Exponential($\frac{1}{2x}$). Calculate $\mathbb{E}[Y|X = x]$ and $\mathbb{E}[Y]$.