Homework 5: finish by 6/9.

Reading: Notes: Chapter 2, 3.

Videos: 3.1 - 3.4, 4.1 - 4.7

Problem 5.1 (**Video 3.3, 3.4**)

Let X be a continuous random variable representing the (exact) lifetime of your TV set, measured in years. A simple model for X is that is an Exponential(λ) random variable. You may assume that your brand of TV has an average lifetime of 20 years.

(a) What is the probability that the TV fails in the first year?

Solution:

Since X is an Exponential(λ) random variable, its average is $E[X] = 1/\lambda$. The average is 20 so $\lambda = 1/20$. The probability that X is less than 1 year is

$$P[X < 1] = F_X(1) = 1 - e^{-1/20} \approx 0.0488 \ .$$

In other words, even though the average lifetime is 20 years, nearly one in 20 TVs will fail within a year!

(b) What is the probability that it lasts more than 5 years?

Solution:

The probability that X is more than 5 years is

$$\mathbb{P}[X > 5] = 1 - \mathbb{P}[X \leq 5] = 1 - F_X(5) = 1 - (1 - e^{-5/20}) = e^{-1/4} \approx 0.7788 \ .$$

(c) Consider the event $B = \{X \geq 10\}$ that your TV has already lasted 10 years. What is the conditional PDF $f_{X|B}(x)$?

Solution:

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}[B]} & x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{P}[B] = \int_{x \in B} f_X(x) \ dx = \int_{10}^{\infty} f_X(x) \ dx = 1 - F_X(10) = e^{-\frac{1}{20} \cdot 10} = e^{-0.5}$$

Thus,

$$f_{X|B}(x) = \begin{cases} \frac{\frac{1}{20} e^{-\frac{1}{20}x}}{e^{-\frac{10}{20}}} & x \geq 10, \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{20} e^{-\frac{1}{20}(x-10)} & x \geq 10, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Let $Y = X - 10$. What is the conditional probability of Y given $B = \{X \geq 10\}$? You can get this by simply transforming $f_{X|B}(x)$ as $f_{Y|B}(y) = f_{X|B}(y + 10)$.

Solution:

From the previous part,

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}[B]} & x \in B, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{\frac{1}{20}e^{-\frac{x}{20}}}{e^{-\frac{10}{20}}} = \frac{1}{20}e^{-\frac{(x-10)}{20}} & x \geq 10, \\ 0 & \text{otherwise.} \end{cases}$$

If we make the variable substitution $Y = X - 10$, then we see that

$$f_{Y|B}(y) = \begin{cases} \frac{1}{20}e^{-\frac{y}{20}} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

is an exponential random variable with the same distribution as X. This is a memoryless property, that says that, conditioned on knowing what time has elapsed so far, the conditional density of the time to go is the same as the original density starting from time 0.

(e) Assume your TV has already lasted 10 years. What is the probability that it fails during the next year?

Solution:

From the previous part, this is the same as the probability that $Y \leq 1$. Since Y is an Exponential($\frac{1}{20}$) RV, this is $F_Y(1) = 1 - e^{-\frac{1}{20}}$.

We will derive this also using two different approaches: one via the conditional PDF from part (c) and another using the definition of conditional probability.

Using the conditional PDF from part (c), we have that

$$\mathbb{P}[X < 11 \,|\, X > 10] = \int_{-\infty}^{11} f_{X|B}(x)\,dx = \int_{10}^{11} \frac{1}{20}e^{-\frac{(x-10)}{20}}\,dx = \left(-e^{-\frac{(x-10)}{20}}\right)\Big|_{10}^{11} = 1-e^{-1/20}$$

Alternatively, define events $A = \{X < 11\}$ and $B = \{X > 10\}$. We are looking for the conditional probability

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}\big[\{X < 11\} \cap \{X > 10\}\big]}{\mathbb{P}[X > 10]} = \frac{\mathbb{P}[10 \leq X < 11]}{\mathbb{P}[X > 10]}.$$

We now evaluate the numerator and denominator,

$$\begin{aligned} \mathbb{P}[10 \leq X < 11] &= \int_{10}^{11} f_X(x)dx = \int_{10}^{11} \frac{1}{20}e^{-x/20}dx \\ &= (-e^{-x/20})\Big|_{10}^{11} = e^{-10/20} - e^{-11/20} = e^{-1/2}\big(1 - e^{-1/20}\big) \end{aligned}$$

Note that $\mathbb{P}[X > 10] = (1 - F_X(10)) = e^{-1/2}$. and find that

$$\mathbb{P}[A|B] = \frac{e^{-1/2}\big(1 - e^{-1/20}\big)}{e^{-1/2}} = 1 - e^{-1/20}.$$

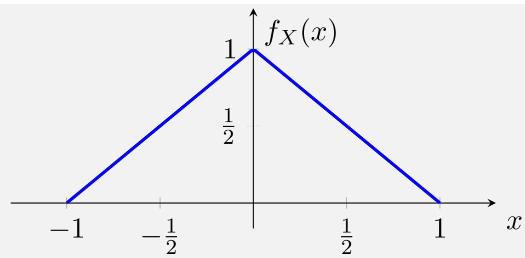
This is exactly the same as the answer for part (a)! The reason this occurs is due to the "memoryless" property of the exponential random variable. In other words, conditioned on the fact that nothing has happened up to time y, it is as if we start out with an entirely new exponential random variable, treating time y as time 0.

Problem 5.2 (**Video 3.1, 3.2, 3.3, 3.4**, **Quick Calculations**)

Calculate each of the requested quantities. **All of these problems are carefully chosen so that they can be completed without integration, so we are expecting exact answers.** For the Gaussian problems, you will sometimes need to lookup values for the standard normal CDF $\Phi(z)$. For example, you could use a lookup table `https://en.wikipedia.org/wiki/Standard_normal_table`, Wolfram Alpha `https://www.wolframalpha.com` with query `normal cdf calculator`, or, in Python, `import scipy.stats as st` followed by `st.norm.cdf(z)` where you should replace `z` with your value.

(a) For $f_X(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$, sketch the PDF, then calculate $\mathbb{E}[X]$ and $\mathbb{P}[|X| > 1/4]$.

Solution:

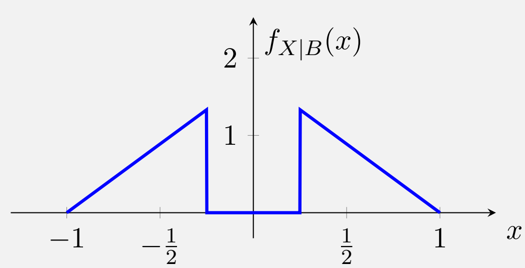


By symmetry, $\mathbb{E}[X] = 0$. For $\mathbb{P}[|X| > 1/4]$, we need to sum up the area of two triangles, each with base $3/4$ and height $3/4$. Therefore, $\mathbb{P}[|X| > 1/4] = 2 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$.

(b) For $f_X(x)$ from part (a), determine and sketch the conditional PDF $f_{X|B}(x)$ given the event $\{|X| > 1/4\}$. Using your sketch, calculate $\mathbb{P}\big[X > 0 \,\big|\, |X| > 1/4\big]$.

Solution:

We have that $f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \frac{16}{9}(1-|x|) & \frac{1}{4} \leq |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$



From the sketch, $\mathbb{P}\big[X > 0 \,\big|\, |X| > 1/4\big] = \frac{1}{2}$ by symmetry.

(c) Let X be Exponential(2). Calculate $\mathbb{E}[X-1]$ and $\mathbb{E}[(X-1)^2]$.

Solution:

$$\begin{aligned}
\mathbb{E}[X-1] &= \mathbb{E}[X] - 1 = \frac{1}{2} - 1 = -\frac{1}{2} \\
\mathbb{E}[(X-1)^2] &= \mathbb{E}[X^2 - 2X + 1] = \mathbb{E}[X^2] - 2\mathbb{E}[X] + 1 \\
&= \mathsf{Var}[X] + \big(\mathbb{E}[X]\big)^2 - 2\mathbb{E}[X] + 1 = \frac{1}{4} + \frac{1}{4} - 2 \cdot \frac{1}{2} + 1 = \frac{1}{2}
\end{aligned}$$

(d) Let X be Uniform$(-2, 3)$. Determine $\mathbb{P}[X - 1 > 0]$ and $\mathbb{P}[X^2 - 1 > 0]$.

Solution:

Note that the PDF of X is has height $1/5$ from $x = -2$ to $x = 3$ and is 0 otherwise. Therefore, all of our calculations can be carried out as areas of rectangles.

$$\begin{aligned}
\mathbb{P}[X-1>0] &= \mathbb{P}[X > 1] = \frac{1}{5} \cdot (3-1) = \frac{2}{5} \\
\mathbb{P}[X^2 - 1 > 0] &= \mathbb{P}[X^2 > 1] = \mathbb{P}[X > 1] + \mathbb{P}[X < -1] \\
&= \frac{1}{5} \cdot (3-1) + \frac{1}{5} \cdot (-1 - (-2)) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}
\end{aligned}$$

(e) Let X be Gaussian$(1, 4)$ and let $Y = 3X - 2$. Determine the mean and variance of Y as well as $\mathbb{P}[Y > 1]$.

Solution:

By the linearity of expectation, $\mathbb{E}[Y] = \mathbb{E}[3X - 2] = 3\mathbb{E}[X] - 2 = 3 \cdot 1 - 2 = 1$.
Using the variance of a linear function, $\mathsf{Var}[Y] = \mathsf{Var}[3X - 2] = 3^2\mathsf{Var}[X] = 9 \cdot 4 = 36$.

$$\mathbb{P}[Y > 1] = 1 - F_Y(1) = 1 - \Phi\left(\frac{1-1}{6}\right) = 1 - \Phi(0) = \frac{1}{2}$$

(f) Let X be Gaussian$(-1, 4)$. Determine (up to two decimal places) the maximum value of a such that $\mathbb{P}[X \le a] \le 0.1$. Determine the minimum value of b such that $\mathbb{P}[X \ge b] \le 0.2$.

Solution:

We know that $\mathbb{P}[X \le a] = F_X(a) = \Phi\big(\frac{a+1}{2}\big)$, and that $F_X(a)$ is an increasing function of a. We need to find where $\Phi(z) = 0.1$ and then solve for a. Using a lookup table, Wolfram Alpha, or Python, we can evaluate $\Phi^{-1}(z)$ to find that $\Phi^{-1}(0.1) = -1.2816$. Solving for a, we get $\frac{a+1}{2} = -1.2816 \implies a = -3.5632$.

We know that $\mathbb{P}[X \ge b] = 1 - \mathbb{P}[X \le b] = 1 - F_X(b) = 1 - \Phi\big(\frac{b+1}{2}\big)$. Using the symmetry of the standard normal CDF $\Phi(z)$, we also have that $\Phi(-z) = 1 - \Phi(z)$. Using a lookup table, Wolfram Alpha, or Python, we can evaluate $\Phi^{-1}(z)$ to find that $\Phi^{-1}(0.2) = -0.8416$, which tells us that $1 - \Phi(0.8416) = 0.2$. Solving for b, we get $\frac{b+1}{2} = 0.8416 \implies b = 0.6832$.

Problem 5.3 (**Video 4.1, 4.2, 4.5, 4.6**, **Lecture Problem**)

Consider the following joint PMF:

$P_{X,Y}(x,y)$		y			
		-2	-1	$+1$	$+2$
x	0	1/12	0	1/6	0
	1	1/3	0	0	1/6
	2	0	1/12	1/12	1/12

(a) Determine the marginal PMFs $P_X(x)$ and $P_Y(y)$.

Solution:

$$P_X(x) = \begin{cases} 1/4 & x = 0, 2 \\ 1/2 & x = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad P_Y(y) = \begin{cases} 5/12 & y = -2 \\ 1/12 & y = -1 \\ 1/4 & y = +1, +2 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Solution:

$$\mathbb{E}[X] = \sum_{x \in S_X} x\, P_X(x) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}[Y] = \sum_{x \in S_Y} y\, P_Y(y) = (-2) \cdot \frac{5}{12} + (-1) \cdot \frac{1}{12} + (+1) \cdot \frac{1}{4}(+2) \cdot \frac{1}{4} = -\frac{1}{6}$$

(c) Calculate $\mathbb{P}[X < Y]$.

Solution:

$$\mathbb{P}[X < Y] = P_{X,Y}(0,1) + P_{X,Y}(0,2) + P_{X,Y}(1,2) = \frac{1}{6} + 0 + \frac{1}{6} = \frac{1}{3}\ .$$

(d) Determine the conditional PMF $P_{X|Y}(x|y)$ and write it out as a table.

Solution:

$P_{X\|Y}(x\|y)$		y			
		-2	-1	$+1$	$+2$
x	0	1/5	0	2/3	0
	1	4/5	0	0	2/3
	2	0	1	1/3	1/3

(e) Calculate $\mathbb{P}[X > 1 | Y = 2]$.

Solution:

$$\mathbb{P}[X > 1|Y = 2] = \sum_{x>1} P_{X|Y}(x|2) = P_{X|Y}(2|2) = \frac{1}{3}$$

(f) Are X and Y independent?

Solution:

No, X and Y are not independent. This is easy to check since there is a zero entry in the joint PMF table. In order for X and Y to be independent, we need that $P_{XY}(x,y) = P_X(x)P_Y(y)$. Therefore, if they were independent a zero entry would imply that either all entries in that column or in that row must also be zero (which is not the case here).

(g) Calculate $\mathbb{E}[2X - 3Y]$.

Solution:

We can use the linearity of expectation combined with our answers from part (b):

$$\mathbb{E}[2X - 3Y] = 2\mathbb{E}[X] - 3\mathbb{E}[Y] = 2 \cdot 1 - 3 \cdot \left(-\frac{1}{6}\right) = \frac{5}{2}$$

(h) Calculate $\mathbb{E}[XY^2]$.

Solution:

$$\begin{aligned}
\mathbb{E}[XY^2] = &\sum_{(x,y)\in S_{X,Y}} xy^2 P_{X,Y}(x,y) \\
&= 0 \cdot (-2)^2 \cdot \frac{1}{12} + 0 \cdot (-1)^2 \cdot 0 + 0 \cdot (+1)^2 \cdot \frac{1}{6} + 0 \cdot (+2)^2 \cdot 0 \\
&+\ 1 \cdot (-2)^2 \cdot \frac{1}{3} + 1 \cdot (-1)^2 \cdot 0 + 1 \cdot (+1)^2 \cdot 0 + 1 \cdot (+2)^2 \cdot \frac{1}{6} \\
&+\ 2 \cdot (-2)^2 \cdot 0 + 2 \cdot (-1)^2 \cdot \frac{1}{12} + 2 \cdot (+1)^2 \cdot \frac{1}{12} + 2 \cdot (+2)^2 \cdot \frac{1}{12} \\
&= 1 \cdot 4 \cdot \frac{1}{3} + 1 \cdot 4 \cdot \frac{1}{6} + 2 \cdot 1 \cdot \frac{1}{2} + 2 \cdot 1 \cdot \frac{1}{12} + 2 \cdot 4 \cdot \frac{1}{12} = 3
\end{aligned}$$

Problem 5.4 (**Video 4.1, 4.2**)

Say that we want to send a bit X from a transmitter to a receiver. We model X as Bernoulli(1/2). The issue is that each transmitted bit may be corrupted (i.e., flipped from a 0 to 1 or a 1 to a 0) with probability 1/4, independently of other bits. One way to overcome this noise is to repeat transmissions several times and take a majority vote among the received bits. We assume that the bit is repeated three times, and let Y be the number of 1's observed at the receiver. It follows that Y given $X = 0$ is Binomial$(3, 1/4)$ and Y given $X = 1$ is Binomial$(3, 3/4)$.

(a) Write out the joint PMF $P_{X,Y}(x,y)$ as a table.

Solution:

We have that

$$P_X(x) = \begin{cases} 1/2 & x = 0, 1 \\ 0 & \text{otherwise} \end{cases} \qquad P_{Y|X}(y|x) = \begin{cases} \binom{3}{y}(\frac{1}{4})^y(\frac{3}{4})^{3-y} & x = 0, y = 0, 1, 2, 3 \\ \binom{3}{y}(\frac{3}{4})^y(\frac{1}{4})^{3-y} & x = 1, y = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Using $P_{X,Y}(x, y) = P_X(x)P_{Y|X}(y|x)$, we get the following joint PMF table

$P_{X,Y}(x,y)$		$y=0$	$y=1$	$y=2$	$y=3$
x	0	27/128	27/128	9/128	1/128
	1	1/128	9/128	27/128	27/128

(b) Determine the marginal PMF $P_Y(y)$.

Solution:

$$P_Y(y) = \sum_{x \in R_X} P_{X,Y}(x, y) = \begin{cases} 7/32 & y = 0 \\ 9/32 & y = 1 \\ 9/32 & y = 2 \\ 7/32 & y = 3 \\ 0 & \text{otherwise} \end{cases}$$

(c) Let $A = \{2, 3\}$ and note that if $Y \in A$, then the majority of the three transmissions result in a 1 observed at the receiver. Calculate $\mathbb{P}[Y \in A]$.

Solution:

$\mathbb{P}[Y \in A] = P_Y(2) + P_Y(3) = 9/32 + 7/32 = 1/2$

(d) Determine $\mathbb{P}[Y \in A|X = 1]$ and $\mathbb{P}[Y \in A|X = 0]$.

Solution:

$$\begin{aligned} \mathbb{P}[Y \in A|X = 1] &= P_{Y|X}(2|1) + P_{Y|X}(3|1) \\ &= \binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1 + \binom{3}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0 = \frac{27}{64} + \frac{27}{64} = \frac{27}{32} \end{aligned}$$

Note that $\mathbb{P}[Y \in A|X = 0] \neq 1 - \mathbb{P}[Y \in A|X = 1]$, i.e., the complement property cannot be applied to the conditioning. Instead, we need to compute this directly

$$\mathbb{P}[Y \in A|X = 0] = P_{Y|X}(2|0) + P_{Y|X}(3|0)$$

$$= \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 + \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 = \frac{1}{64} + \frac{9}{64} = \frac{5}{32}$$

(e) Calculate the probability that this majority vote is correct, $\mathbb{P}[X = 1|Y \in A]$, and incorrect, $\mathbb{P}[X = 0|Y \in A]$.

Solution:

By Bayes' Rule,

$$\begin{aligned}
\mathbb{P}[X = 1|Y \in A] &= \frac{\mathbb{P}[Y \in A|X = 1]\mathbb{P}[X = 1]}{\mathbb{P}[Y \in A]} = \frac{\mathbb{P}[Y \in A|X = 1]P_X(1)}{\mathbb{P}[Y \in A]} \\
&= \frac{\frac{27}{32} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{27}{32} \,.
\end{aligned}$$

Here, we can use the complement property, $\mathbb{P}[X = 0|Y \in A] = 1 - \mathbb{P}[X = 1|Y \in A] = \dfrac{5}{32}$.

Problem 5.5 (**Video 4.3, 4.5, 4.6**, **Lecture Problem**)

The continuous random variables X and Y have joint PDF

$$f_{XY}(x, y) = \begin{cases} cx^2y & -1 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of the constant c that will satisfy the normalization property. Set c to this value for the remainder of the problem.

Solution:

Recall that you will get full credit for writing down an integral with the correct limits. The relevant integrals are shown in blue below. You can evaluate the integral if you are interested (and the solution below does) but this will not affect your grade.

The joint PDF must integrate to 1 over the range of possible values for X and Y. First, let's evaluate the integral of the joint PDF:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy &= \int_{-1}^{1}\int_{0}^{2} cx^2y\,dy\,dx \\
&= \int_{-1}^{1} \left(\frac{c}{2}x^2y^2\right)\Big|_0^2\,dx \\
&= \int_{-1}^{1} 2cx^2\,dx \\
&= \left(\frac{2c}{3}x^3\right)\Big|_{-1}^{1} = \frac{4c}{3} \,.
\end{aligned}$$

Therefore, we should set $c = 3/4$ to satisfy the normalization property.

(b) Calculate $P\big[X \leq 0, Y \leq 1\big]$.

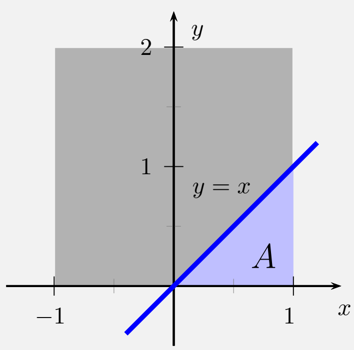
Solution:

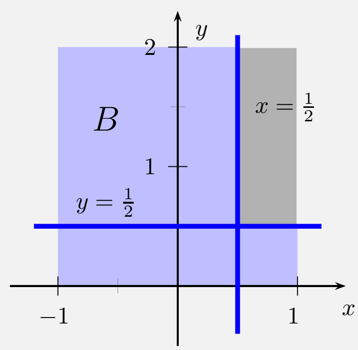
$$\begin{aligned}
P\big[X \le 0,\, Y \le 1\big] &= \int_{-\infty}^{0}\int_{-\infty}^{1} f_{XY}(x,y)\,dy\,dx \\
&= \int_{-1}^{0}\int_{0}^{1} \frac{3}{4}x^2 y\,dy\,dx \\
&= \int_{-1}^{0} \left(\frac{3}{8}x^2y^2\right)\Big|_0^1\,dx \\
&= \int_{-1}^{0} \frac{3}{8}x^2\,dx \\
&= \left(\frac{1}{8}x^3\right)_{-1}^{0} = \frac{1}{8}
\end{aligned}$$

(c) What is the probability that Y is less than X?

Solution:

We are interested in the event $\{Y < X\}$. This corresponds to the event $(X,Y) \in A$ where $A = \big\{(x,y) : 0 \le x \le 1,\, 0 \le y < x\big\}$ is the set of possible pairs in the range $S_{XY} = \big\{(x,y) : -1 \le x \le 1,\, 0 \le y \le 2\big\}$ for which $y < x$. The set A is easier to write down if you draw a sketch first. See below for an illustration of the integration region.





$$P[Y < X] = \int_0^1\int_0^x \frac{3}{4}x^2y\,dy\,dx = \int_0^1 \left(\frac{3}{8}x^2y^2\right)\Big|_0^x dx = \int_0^1 \frac{3}{8}x^4\,dx = \left(\frac{3}{40}x^5\right)\Big|_0^1 = \frac{3}{40}$$

Note that you can change the order of integration, but the limits need to change too:

$$P[Y < X] = \int_0^1\int_y^1 \frac{3}{4}x^2y\,dx\,dy\ .$$

(d) Calculate $P\big[\min(X,Y) \le \frac{1}{2}\big]$.

Solution:

We are interested in the event $\{\min(X,Y) \le \frac{1}{2}\}$. This corresponds to the event $(X,Y) \in B$ where

$$B = \Big\{(x,y) : -1 \le x \le \frac{1}{2},\, 0 \le y < 2\Big\} \bigcup \Big\{(x,y) : -1 \le x \le 1,\, 0 \le y < \frac{1}{2}\Big\}$$

is the set of possible pairs in the range $R_{XY} = \big\{(x,y) : -1 \le x \le 1,\, 0 \le y \le 2\big\}$ for which $y < x$. The set B is easier to write down if you draw a sketch first. See above for an illustration of the integration region.

There are three methods to evaluate $P[(X,Y) \in B]$. First, since B consists of a union of two regions, we could use the inclusion-exclusion principle. Second, we could partition B into two mutually exclusive regions. Third, we could recognize that the complement of B is just a rectangle $B^c = \big\{(x,y) : \frac{1}{2} < x \le 1,\, \frac{1}{2} < y < 2\big\}$. The last method is the easiest.

$$\begin{aligned} P[(X,Y) \in B^c] &= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{2} \frac{3}{4} x^2 y \, dy \, dx \\ &= \int_{\frac{1}{2}}^{1} \Big(\frac{3}{8} x^2 y^2\Big)\Big|_{\frac{1}{2}}^{2} dx = \int_{\frac{1}{2}}^{1} \frac{45}{32} x^2 \, dx = \Big(\frac{15}{32} x^3\Big)\Big|_{\frac{1}{2}}^{1} = \frac{105}{256} \,. \end{aligned}$$

Therefore, we have $P\Big[\min(X,Y) \le \dfrac{1}{2}\Big] = 1 - P\big[(X,Y) \in B^c\big] = \dfrac{151}{256}$.

(e) Calculate the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution:

$$\begin{aligned} f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy &= \begin{cases} \displaystyle\int_0^2 \frac{3}{4} x^2 y \, dy & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \Big(\dfrac{3}{8} x^2 y^2\Big)\Big|_0^2 & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \dfrac{3}{2} x^2 & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \begin{cases} \int_{-1}^{1} \frac{3}{4}x^2 y\,dx & 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left(\frac{1}{4}x^3 y\right)\Big|_{-1}^{1} & 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{2}y & 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(f) Compute the expected values $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Solution:

Since the PDF of X is symmetric about 0, we know that $\mathbb{E}[X] = 0$. We can double-check by taking the usual integral:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-1}^{1} \frac{3}{2}x^3\,dx = \left(\frac{3}{8}x^4\right)\Big|_{-1}^{1} = 0\ .$$

The expected value of Y is less obvious and requires an integral,

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{0}^{2} \frac{1}{2}y^2\,dy = \left(\frac{1}{6}y^3\right)\Big|_{0}^{2} = \frac{4}{3}\ .$$

(g) Are X and Y independent?

Solution:

Yes, X and Y are independent since $f_{XY}(x,y) = f_X(x)f_Y(y)$.

(h) Compute $\mathbb{E}[X^4 Y]$.

Solution:

Since X and Y are independent, we can factor this expected value as $\mathbb{E}[X^4 Y] = \mathbb{E}[X^4]\mathbb{E}[Y]$. We already have $\mathbb{E}[Y]$ from part (f) and just need to calculate $\mathbb{E}[X^4]$.

$$\mathbb{E}[X^4] = \int_{-\infty}^{\infty} x^4 f_X(x)\,dx = \int_{-1}^{1} \frac{3}{2}x^6\,dx = \left(\frac{3}{14}x^7\right)\Big|_{-1}^{1} = \frac{3}{7}\ .$$

Therefore, $\mathbb{E}[X^4 Y] = \frac{3}{7} \cdot \frac{4}{3} = \frac{4}{7}$.