EK381 Probability, Statistics, and Data Science for Engineers
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Boston University
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Homework 5: finish by 6/9.

Reading: Notes: Chapter 2, 3. **Videos:** 3.1 - 3.4, 4.1 - 4.7

Problem 5.1 (**Video 3.3, 3.4**)

Let X be a continuous random variable representing the (exact) lifetime of your TV set, measured in years. A simple model for X is that is an Exponential(λ) random variable. You may assume that your brand of TV has an average lifetime of 20 years.

(a) What is the probability that the TV fails in the first year?

(b) What is the probability that it lasts more than 5 years?

(c) Consider the event $B = \{X \geq 10\}$ that your TV has already lasted 10 years. What is the conditional PDF $f_{X|B}(x)$?

(d) Let $Y = X - 10$. What is the conditional probability of Y given $B = \{X \geq 10\}$? You can get this by simply transforming $f_{X|B}(x)$ as $f_{Y|B}(y) = f_{X|B}(y + 10)$.

(e) Assume your TV has already lasted 10 years. What is the probability that it fails during the next year?

Problem 5.2 (**Video 3.1, 3.2, 3.3, 3.4**, **Quick Calculations**)

Calculate each of the requested quantities. **All of these problems are carefully chosen so that they can be completed without integration, so we are expecting exact answers.** For the Gaussian problems, you will sometimes need to lookup values for the standard normal CDF $\Phi(z)$. For example, you could use a lookup table `https://en.wikipedia.org/wiki/Standard_normal_table`, Wolfram Alpha `https://www.wolframalpha.com` with query `normal cdf calculator`, or, in Python, `import scipy.stats as st` followed by `st.norm.cdf(z)` where you should replace `z` with your value.

(a) For $f_X(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$, sketch the PDF, then calculate $\mathbb{E}[X]$ and $\mathbb{P}[|X| > 1/4]$.

(b) For $f_X(x)$ from part (a), determine and sketch the conditional PDF $f_{X|B}(x)$ given the event $\{|X| > 1/4\}$. Using your sketch, calculate $\mathbb{P}\big[X > 0 \,\big|\, |X| > 1/4\big]$.

(c) Let X be Exponential(2). Calculate $\mathbb{E}[X - 1]$ and $\mathbb{E}[(X - 1)^2]$.

(d) Let X be Uniform$(-2, 3)$. Determine $\mathbb{P}[X - 1 > 0]$ and $\mathbb{P}[X^2 - 1 > 0]$.

(e) Let X be Gaussian$(1, 4)$ and let $Y = 3X - 2$. Determine the mean and variance of Y as well as $\mathbb{P}[Y > 1]$.

(f) Let X be Gaussian$(-1, 4)$. Determine (up to two decimal places) the maximum value of a such that $\mathbb{P}[X \leq a] \leq 0.1$. Determine the minimum value of b such that $\mathbb{P}[X \geq b] \leq 0.2$.

Problem 5.3 (**Video 4.1, 4.2, 4.5, 4.6**, **Lecture Problem**)

Consider the following joint PMF:

$P_{X,Y}(x,y)$		$y=-2$	-1	$+1$	$+2$
x	0	1/12	0	1/6	0
	1	1/3	0	0	1/6
	2	0	1/12	1/12	1/12

(a) Determine the marginal PMFs $P_X(x)$ and $P_Y(y)$.

(b) Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(c) Calculate $\mathbb{P}[X < Y]$.

(d) Determine the conditional PMF $P_{X|Y}(x|y)$ and write it out as a table.

(e) Calculate $\mathbb{P}[X > 1|Y = 2]$.

(f) Are X and Y independent?

(g) Calculate $\mathbb{E}[2X - 3Y]$.

(h) Calculate $\mathbb{E}[XY^2]$.

Problem 5.4 (**Video 4.1, 4.2**)

Say that we want to send a bit X from a transmitter to a receiver. We model X as Bernoulli(1/2). The issue is that each transmitted bit may be corrupted (i.e., flipped from a 0 to 1 or a 1 to a 0) with probability 1/4, independently of other bits. One way to overcome this noise is to repeat transmissions several times and take a majority vote among the received bits. We assume that the bit is repeated three times, and let Y be the number of 1's observed at the receiver. It follows that Y given $X = 0$ is Binomial$(3, 1/4)$ and Y given $X = 1$ is Binomial$(3, 3/4)$.

(a) Write out the joint PMF $P_{X,Y}(x,y)$ as a table.

(b) Determine the marginal PMF $P_Y(y)$.

(c) Let $A = \{2, 3\}$ and note that if $Y \in A$, then the majority of the three transmissions result in a 1 observed at the receiver. Calculate $\mathbb{P}[Y \in A]$.

(d) Determine $\mathbb{P}[Y \in A|X = 1]$ and $\mathbb{P}[Y \in A|X = 0]$.

(e) Calculate the probability that this majority vote is correct, $\mathbb{P}[X = 1|Y \in A]$, and incorrect, $\mathbb{P}[X = 0|Y \in A]$.

Problem 5.5 (**Video 4.3, 4.5, 4.6**, **Lecture Problem**)

The continuous random variables X and Y have joint PDF

$$f_{XY}(x,y) = \begin{cases} cx^2y & -1 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of the constant c that will satisfy the normalization property. Set c to this value for the remainder of the problem.

(b) Calculate $P\big[X \leq 0, Y \leq 1\big]$.

(c) What is the probability that Y is less than X?

(d) Calculate $P\big[\min(X, Y) \leq \frac{1}{2}\big]$.

(e) Calculate the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(f) Compute the expected values $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(g) Are X and Y independent?

(h) Compute $\mathbb{E}[X^4 Y]$.