EK381 Probability, Statistics, and Data Science for Engineers — Boston University
Prof Carruthers — Summer 2025

Homework 1. Finish by 5/22.

Reading: Notes: Chapter 1. **Videos:** 1.1, 1.2, 1.3, 1.4

Gradescope Uploading Instructions: Please be sure to assign problems to PDF pages when you are uploading to Gradescope. You may be docked points otherwise. Detailed instructions are available in the Homework folder on Blackboard.

Lecture Problems: To help make up for the time that you spend outside of lecture watching videos, some homework problems will be solved during lecture time. These will be denoted throughout the semester by the phrase "**Lecture Problem**."

Quick Calculations: Every homework will have a problem that focuses on quick calculations to help you get familiar with the mechanics of the concepts introduced that week. This will also help prepare you for exams, which will also include a similar problem.

Video-Problem Correspondence For each problem, we will indicate the videos that you should watch in advance.

Problem 1.1 (**Video 1.1**, **Lecture Problem**) Let A, B, C be three events in a sample space S. Each of the statements below describes an event built from events A, B, and C. For each statement, express the resulting event in terms of the events A, B, and C using only the complement, union, and intersection operations. Also, for each statement, draw an appropriate Venn diagram and shade the resulting event.

(a) at least one of the events A, B occurs but C does not occur

(b) at most one of the events A, B occurs;

(c) neither of the events A, B occurs;

(d) events A, B occur but C does not occur;

(e) either event A occurs or, if not, then B also does not occur.

Solution:

Note that there are many logical ways to write the same thing, so your answers may differ:

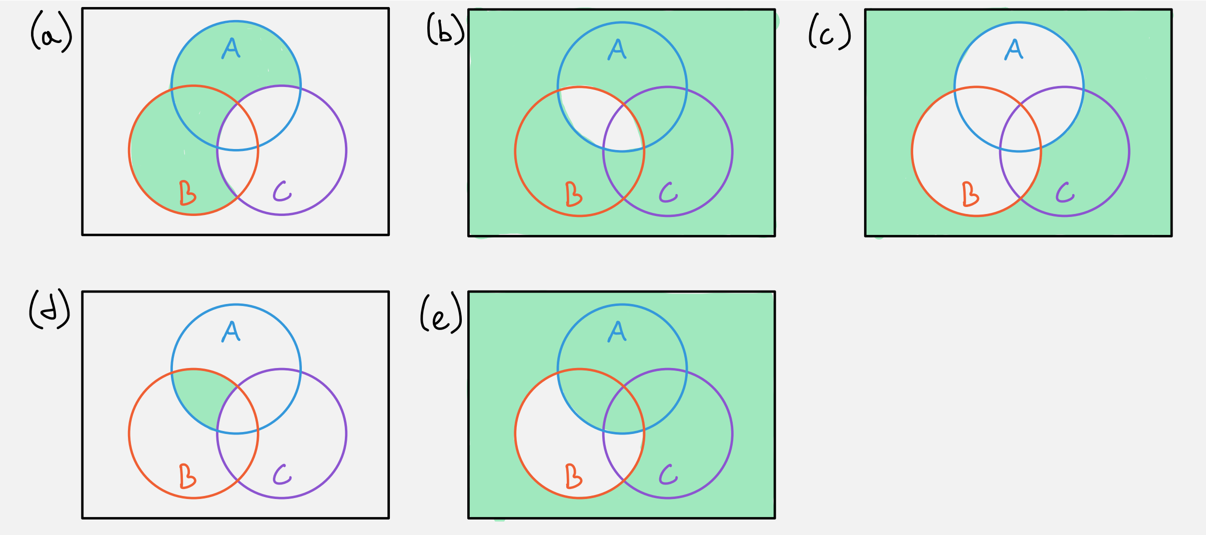
(a) $(A \cup B) \cap C^c$

(b) $(A \cap B^c) \cup (B \cap A^c) \cup (A^c \cap B^c)$

(c) $(A \cup B)^c$

(d) $A \cap B \cap C^c$

(e) $A \cup (A^c \cap B^c)$



Problem 1.2 (**Video 1.2**, **Lecture Problem**) If a sample space Ω consists of a *finite* number of outcomes, then it is possible to directly assign each outcome its own probability. In this special case, the probability of any event can be calculated by adding up the probabilities of its individual outcomes. Specifically, if $E = \{s_1, s_2, \dots, s_m\}$, then

$$\mathbb{P}[E] = \sum_{i=1}^{m} \mathbb{P}[s_i] \ .$$

Additionally, if all outcomes are equally likely, this formula simplifies to

$$\mathbb{P}[E] = \frac{\text{\# of outcomes in } E}{\text{\# of outcomes in } \Omega} \ .$$

(a) Say that we roll a four-sided die with faces $\{1,2,3,4\}$ followed by a five-sided die with faces $\{1,2,3,4,5\}$. All outcomes are equally likely. Write down the sample space Ω for this experiment as well as the probability for each outcome.

(b) Calculate the probability of both rolls being odd.

(c) Calculate the probability that the sum of the rolls is 6.

(d) Calculate the probability that the sum of the rolls is 6 *or* both rolls are odd.

(e) Verify your answer for part (d) using the inclusion-exclusion principle.

Solution:

(a) The sample space consists of 20 outcomes

$$\Omega = \big\{(i,j) : i \in \{1,2,3,4\},\ j \in \{1,2,3,4,5\}\big\} \ .$$

Each outcome has probability $\frac{1}{20}$.

(b) $$\begin{aligned} A &= \{\text{both rolls odd}\} \\ &= \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5)\} \\ \mathbb{P}[A] &= \frac{6}{20} = \frac{3}{10} \end{aligned}$$

(c) $$\begin{aligned} B &= \{\text{sum of rolls is } 6\} \\ &= \{(1,5),(2,4),(3,3),(4,2)\} \\ \mathbb{P}[B] &= \frac{4}{20} = \frac{1}{5} \end{aligned}$$

(d) $$\begin{aligned} A \cup B &= \{\text{both rolls odd } \textit{or} \text{ sum of rolls is } 6\} \\ &= \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(2,4),(4,2)\} \\ \mathbb{P}[A \cup B] &= \frac{8}{20} = \frac{2}{5} \end{aligned}$$

(e) $$\begin{aligned} A \cap B &= \{\text{both rolls odd } \textit{and} \text{ sum of rolls is } 6\} \\ &= \{(1,5),(3,3)\} \\ \mathbb{P}[A \cap B] &= \frac{2}{20} = \frac{1}{10} \\ \mathbb{P}[A \cup B] &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \frac{6+4-2}{20} = \frac{8}{20} = \frac{2}{5} \end{aligned}$$

Problem 1.3 (**Video 1.3, 1.4**, **Lecture Problem**) Thor is trying to determine Loki's recent whereabouts. His approach is to go to a random planet and ask the local leader, "Has Loki visited this planet recently?" to which they will answer either "Yes" or "No." There is one problem: some of the planets are secretly controlled by Loki and the leaders will lie to protect him.

The probability that a planet is controlled by Loki is $1/3$. The probability that Loki has visited a planet recently is $1/2$ if he controls it and $1/4$ if he does not control it. If the planet is controlled by Loki, the leader will lie with probability $3/4$. If the planet is not controlled by Loki, a leader will lie with probability $1/5$ (for their own reasons that have nothing to do with Loki).

(a) Thor picks a random planet and asks its leader if Loki has visited recently. Draw and label a tree diagram of all the probabilities and conditional probabilities leading up to the leader's answer. Specifically, let C be the event that Loki controls the planet, V be the event that Loki has visited this planet recently, and Y be the event that the leader answers "Yes." (You can think of the answer "No" as Y^c.) Draw the tree with the sample space Ω at the root, followed by C, then V, and then Y.

(b) Determine the probability that Loki does not control the planet *and* Loki has visited recently *and* the leader answers "Yes," $\mathbb{P}[C^c \cap V \cap Y]$.

(c) Determine the probability that the leader answers "Yes" given that Loki has visited recently, $\mathbb{P}[Y|V]$.

(d) Determine the probability that the leader answers "Yes," $\mathbb{P}[Y]$.

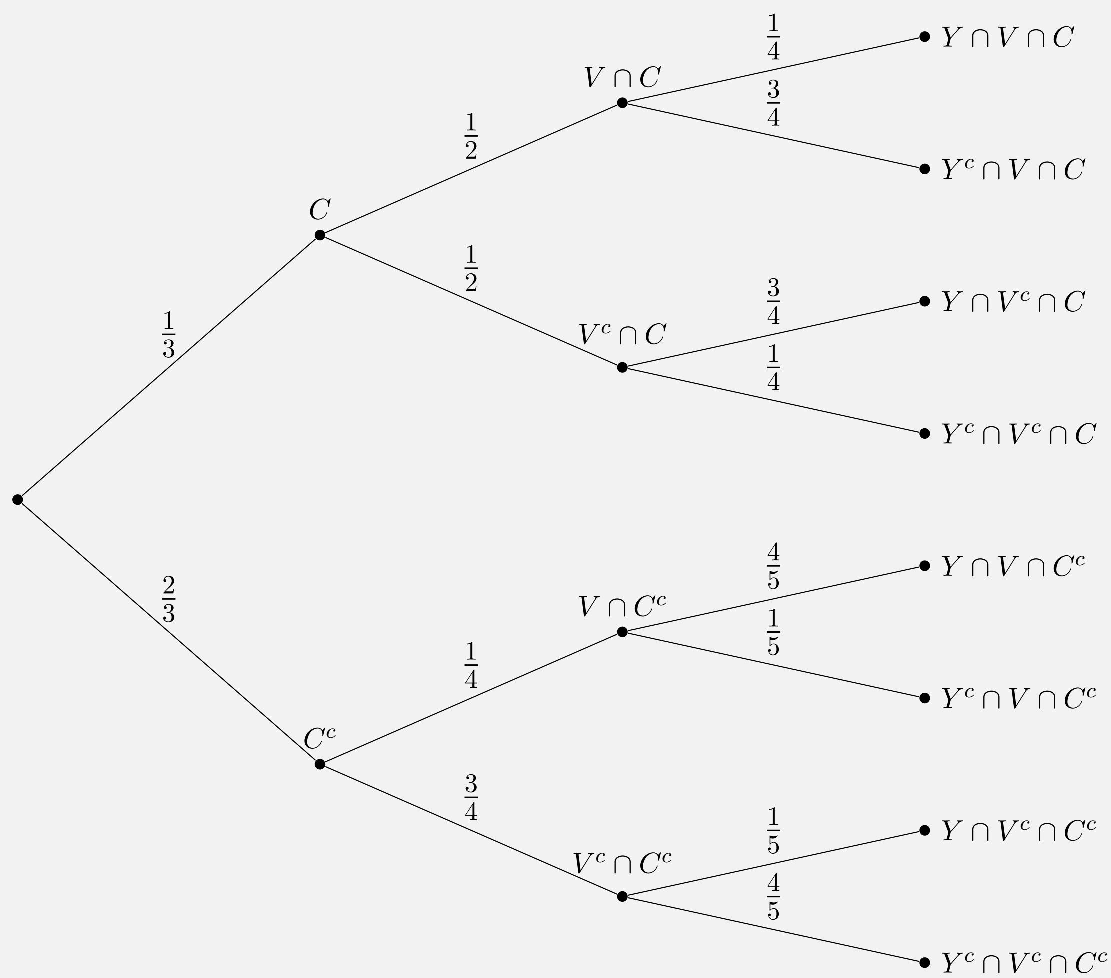
(e) Determine the probability that Loki has visited recently given that the leader answers "Yes," $\mathbb{P}[V|Y]$.

(f) After some careful observation, Thor manages to identify a planet that Loki does not control. Given that the leader answers "Yes" on this planet, what is the probability that Loki has visited recently, $\mathbb{P}[V|Y \cap C^c]$?

Solution:

For parts (b) through (f), we will use the hint and a few equations. It is also possible to work directly with the tree diagram as we did in class, but this is not shown in this solution.

(a)



(b) Following the hint, we can use the Multiplication Rule to get

$$\mathbb{P}[C^c \cap V \cap Y] = \mathbb{P}[C^c]\mathbb{P}[V|C^c]\mathbb{P}[Y|C^c \cap V] = \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} = \frac{2}{15}\ .$$

(c) We know from the definition of conditional probability that $\mathbb{P}[Y|V] = \dfrac{\mathbb{P}[Y \cap V]}{\mathbb{P}[V]}$.

Next, we should solve for $\mathbb{P}[Y \cap V]$:

$$\begin{aligned}
\mathbb{P}[Y \cap V] &\overset{(i)}{=} \mathbb{P}[Y \cap V|C]\mathbb{P}[C] + \mathbb{P}[Y \cap V|C^c]\mathbb{P}[C^c] \\
&\overset{(ii)}{=} \mathbb{P}[Y|C \cap V]P[V|C]\mathbb{P}[C] + \mathbb{P}[Y|C^c \cap V]\mathbb{P}[V|C^c]\mathbb{P}[C^c] \\
&= \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} \\
&= \frac{1}{24} + \frac{2}{15} \\
&= \frac{7}{40}
\end{aligned}$$

where in (i) we used the Law of Total Probability treating C and C^c as a partition and in (ii) we used the Multiplication Rule (twice). Now, we need to solve for $\mathbb{P}[V]$ using the Law of Total Probability, treating C and C^c as a partition:

$$\begin{aligned}
\mathbb{P}[V] &= \mathbb{P}[V|C]\mathbb{P}[C] + \mathbb{P}[V|C^c]\mathbb{P}[C^c] \\
&= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} \\
&= \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{3}
\end{aligned}$$

Plugging back into the definition of conditional probability, we get

$$\mathbb{P}[Y|V] = \frac{\mathbb{P}[Y \cap V]}{\mathbb{P}[V]} = \frac{\frac{7}{40}}{\frac{1}{3}} = \frac{21}{40} .$$

(d) We can use the Law of Total Probability using V and V^c as a partition to get

$$\mathbb{P}[Y] = \mathbb{P}[Y|V]\mathbb{P}[V] + \mathbb{P}[Y|V^c]\mathbb{P}[V^c] .$$

From part (c), we know that $\mathbb{P}[V] = 1/3$ and thus $\mathbb{P}[V^c] = 1 - \mathbb{P}[V] = 1 - 1/3 = 2/3$. From part (c), we know that $\mathbb{P}[Y|V] = 21/40$. We just need $\mathbb{P}[Y|V^c]$, which we can obtain by repeating the steps from part (c),

$$\begin{aligned}
\mathbb{P}[Y \cap V^c] &\overset{(i)}{=} \mathbb{P}[Y \cap V^c|C]\mathbb{P}[C] + \mathbb{P}[Y \cap V^c|C^c]\mathbb{P}[C^c] \\
&\overset{(ii)}{=} \mathbb{P}[Y|C \cap V^c]P[V^c|C]\mathbb{P}[C] + \mathbb{P}[Y|C^c \cap V^c]\mathbb{P}[V^c|C^c]\mathbb{P}[C^c] \\
&= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \\
&= \frac{1}{8} + \frac{1}{10} \\
&= \frac{9}{40}
\end{aligned}$$

where in (i) we used the Law of Total Probability treating C and C^c as a partition and in (ii) we used the Multiplication Rule (twice). Plugging into the definition of conditional probability, we get

$$\mathbb{P}[Y|V^c] = \frac{\mathbb{P}[Y \cap V^c]}{\mathbb{P}[V^c]} = \frac{\frac{9}{40}}{\frac{2}{3}} = \frac{27}{80} \ .$$

(Note that, in general, $\mathbb{P}[A|B] \neq 1 - \mathbb{P}[A|B^c]$.)

Finally, we can go back to our first equation and evaluate terms:

$$\mathbb{P}[Y] = \mathbb{P}[Y|V]\mathbb{P}[V] + \mathbb{P}[Y|V^c]\mathbb{P}[V^c] = \frac{21}{40} \cdot \frac{1}{3} + \frac{27}{80} \cdot \frac{2}{3} = \frac{2}{5} \ .$$

(e) We already have $\mathbb{P}[Y|V]$ and $\mathbb{P}[V]$ from part (c) as well as $\mathbb{P}[Y]$ from part (d). Therefore, we can use Bayes' Rule to "flip" the conditioning:

$$\mathbb{P}[V|Y] = \frac{\mathbb{P}[Y|V]\mathbb{P}[V]}{\mathbb{P}[Y]} = \frac{\frac{21}{40} \cdot \frac{1}{3}}{\frac{2}{5}} = \frac{7}{16} \ .$$

(f) From the definition of conditional probability, $\mathbb{P}[V|Y \cap C^c] = \dfrac{\mathbb{P}[V \cap Y \cap C^c]}{\mathbb{P}[Y \cap C^c]}$. We already have $\mathbb{P}[V \cap Y \cap C^c] = 2/15$ from part (b). Notice that we can decompose $Y \cap C^c$ into the union of two mutually exclusive events $Y \cap C^c \cap V$ and $Y \cap C^c \cap V^c$,

$$Y \cap C^c = \big(Y \cap C^c \cap V\big) \cup \big(Y \cap C^c \cap V^c\big) \ ,$$

which means that we can use the Additivity Property to get that

$$\mathbb{P}[Y \cap C^c] = \mathbb{P}[Y \cap C^c \cap V] + \mathbb{P}[Y \cap C^c \cap V^c] \ .$$

We already know the first term is $2/15$ from part (b). For the second term, we can use the Multiplication Rule,

$$\mathbb{P}[Y \cap C^c \cap V^c] = \mathbb{P}[C^c]\mathbb{P}[V^c|C^c]\mathbb{P}[Y|V^c \cap C^c] = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} = \frac{1}{10} \ .$$

Plugging back in, we get $\mathbb{P}[Y \cap C^c] = \frac{2}{15} + \frac{1}{10} = \frac{7}{30}$. Finally, we plug back into the first conditional probability equation to get

$$\mathbb{P}[V|Y \cap C^c] = \frac{\mathbb{P}[V \cap Y \cap C^c]}{\mathbb{P}[Y \cap C^c]} = \frac{\frac{2}{15}}{\frac{7}{30}} = \frac{4}{7} \ .$$

Problem 1.4 (**Video 1.3, 1.4**) Your company uses a diverse set 3 manufacturers ($M = A, B, C$) of the same part to balance cost, quality, and availability. The probability that a product fails (denoted by event F) given that it was produced by manufacturer M is shown in the table below.

Manufacturer $M =$	A	B	C
$\mathbb{P}(F|M) =$	$\frac{2}{10}$	0	$\frac{4}{10}$

The proportion of parts that you order from manufacturer A is $\frac{55}{100}$, from B is $\frac{5}{100}$, and the remaining are from C. You randomly choose a part from your entire inventory and check if it works or fails. Assume that all parts in the inventory are equally likely to be chosen.

(a) What is the probability that the randomly-chosen part is manufactured by A *and* it works.

(b) What is the probability that the randomly-chosen part works?

(c) Given that the randomly-chosen part works, what is the probability that it is manufactured by A?

Solution:

First note that the event "part works" is the complement of the event "part fails", i.e., "part works" $= F^c$.

(a) This is an application of the Multiplication Rule:

$$\mathbb{P}[A \cap F^c] = \mathbb{P}[A]\mathbb{P}[F^c|A] = \mathbb{P}[A](1 - \mathbb{P}[F|A]) = \frac{55}{100} \cdot \left(1 - \frac{2}{10}\right) = \frac{440}{1000} = 0.44.$$

(b) First, we need to determine the probability of choosing a part manufactured by E via the complement property:

$$\mathbb{P}[C] = 1 - \mathbb{P}[A] - \mathbb{P}[B] = 1 - \frac{55}{100} - \frac{5}{100} = \frac{40}{100}\ .$$

Now, this is an application of the Law of Total Probability where the set of manufacturers act as a partition:

$$\begin{aligned}\mathbb{P}[F^c] &= \mathbb{P}[F^c|A]\,\mathbb{P}[A] + \mathbb{P}[F^c|B]\,\mathbb{P}[B] + \mathbb{P}[F^c|C]\,\mathbb{P}[C]\\ &= \frac{8}{10} \cdot \frac{55}{100} + \frac{10}{10} \cdot \frac{5}{100} + \frac{6}{10} \cdot \frac{40}{100} = \frac{730}{1000} = 0.73\ .\end{aligned}$$

(c) Since we need to "flip" the conditioning, this is just an application of Bayes' Rule,

$$\mathbb{P}[A|F^c] = \frac{\mathbb{P}[F^c|A]\mathbb{P}[A]}{\mathbb{P}[F^c]} = \frac{\frac{8}{10} \cdot \frac{55}{100}}{\frac{730}{1000}} = \frac{44}{73} \approx 0.6027.$$

Important observation and a potential source of confusion: $\mathbb{P}[F^c|M] + \mathbb{P}[F|M] = 1$ for all $M = A, B, C$. This is the law of total probability *with conditioning* and observe that we are keeping the conditioning event M fixed in the summation. On the other hand, $\mathbb{P}[F|A] + \mathbb{P}[F|B] + \mathbb{P}[F|C] = \frac{2}{10} + 0 + \frac{4}{10} = \frac{6}{10} \neq 1$. Thus, if we sum conditional probabilities across *different conditioning events*, the sum may not add up to one.

Problem 1.5 (**Video 1.1, 1.2, 1.3, 1.4**, **Quick Calculations**) Determine each of the requested quantities.

(a) Let $\Omega = \{1, 2, 3\}$. Assume that $\mathbb{P}[\{1, 2\}] = 1/3$ and $\mathbb{P}[\{2, 3\}] = 5/6$. What is $\mathbb{P}[\{2\}]$?

(b) Assume that $\mathbb{P}[A \cap B] = 0.1$ and $\mathbb{P}[A \cup B^{\mathsf{c}}] = 0.8$. What is $\mathbb{P}[B]$?

(c) Let A and B be events with $\mathbb{P}[A \cap B] = 5/8$ and $\mathbb{P}[A^{\mathsf{c}} \cap B] = 1/8$. Calculate $\mathbb{P}[B]$ and $\mathbb{P}[A|B]$.

(d) Let $\Omega = \{1, 2, 3, 4, 5\}$. Assume that $\mathbb{P}[\{1\}] = 0.1$, $\mathbb{P}[\{2\}] = 0.2$, and $\mathbb{P}[\{3\}] = 0.2$. What is $\mathbb{P}[\{3, 4, 5\}]$?

(e) In Metropolis, the probability that a random citizen has Covid is 0.1 (determined via high-quality surveillance testing). LexCorp is selling a test that returns false negatives with conditional probability 0.2 and false positives with conditional probability 0.05. Given that a citizen tests negative, what is the conditional probability that they actually have Covid? If a citizen tests positive, what is the conditional probability that they actually have Covid?

Solution:

(a) First, note that, by the complement property, $\mathbb{P}[\{3\}] = \mathbb{P}[\{1, 2\}^{\mathsf{c}}] = 1 - \mathbb{P}[\{1, 2\}] = 1 - 1/3 = 2/3$. Second, by the additivity axiom, we must have that $\mathbb{P}[\{2, 3\}] = \mathbb{P}[\{2\}] + \mathbb{P}[\{3\}] = 5/6$. Therefore, $\mathbb{P}[\{2\}] = 5/6 - \mathbb{P}[\{3\}] = 5/6 - 2/3 = 1/6$.

(b) Using De Morgan's Laws, we know that $(A \cup B^{\mathsf{c}})^{\mathsf{c}} = A^{\mathsf{c}} \cap B$. Therefore, by the complement property, $\mathbb{P}[A^{\mathsf{c}} \cap B] = 1 - \mathbb{P}[(A^{\mathsf{c}} \cap B)^{\mathsf{c}}] = 1 - \mathbb{P}[A \cup B^{\mathsf{c}}] = 1 - 0.8 = 0.2$. Next, by the additivity axiom, we have that $\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^{\mathsf{c}} \cap B] = 0.1 + 0.2 = 0.3$, since $A \cap B$ and $A^{\mathsf{c}} \cap B$ are mutually exclusive.

(c) $$\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^{\mathsf{c}} \cap B] = \frac{5}{8} + \frac{1}{8} = \frac{3}{4}$$

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{6}$$

(d) First, note that by the additivity axiom, $\mathbb{P}[\{1, 2\} = \mathbb{P}[\{1\}] + \mathbb{P}[\{2\}] = 0.1 + 0.2 = 0.3$. Second, by the complement property, we have that $\mathbb{P}[\{3, 4, 5\}] = 1 - \mathbb{P}[\{3, 4, 5\}^{\mathsf{c}}] = 1 - \mathbb{P}[\{1, 2\}] = 1 - 0.3 = 0.7$. (The value of $\mathbb{P}[\{3\}]$ is not needed to solve the problem.)

(e) Let C be the event that a citizen has Covid and N be the event that the LexCorp test comes back negative. We know that $\mathbb{P}[C] = 0.1$, $\mathbb{P}[N|C] = 0.2$ (false negative), and $\mathbb{P}[N^{\mathsf{c}}|C^{\mathsf{c}}] = 0.05$ (false positive). By the complement property, we can infer that $\mathbb{P}[C^{\mathsf{c}}] = 0.9$, $\mathbb{P}[N^{\mathsf{c}}|C] = 0.8$ (true positive), and $\mathbb{P}[N|C^{\mathsf{c}}] = 0.95$ (true negative). The probability that a citizen has Covid given that their test is negative is the conditional probability $\mathbb{P}[C|N]$. Since we want to "flip" the conditioning from the known quantity $\mathbb{P}[N|C]$, we should use Bayes' Rule, along with the Law of Total Probability to expand the denominator:

$$\begin{aligned}
\mathbb{P}[C|N] &= \frac{\mathbb{P}[N|C]\,\mathbb{P}[C]}{\mathbb{P}[N|C]\,\mathbb{P}[C] + \mathbb{P}[N|C^{\mathsf{c}}]\,\mathbb{P}[C^{\mathsf{c}}]} \\
&= \frac{0.2 \cdot 0.1}{0.2 \cdot 0.1 + 0.95 \cdot 0.9} \\
&= \frac{4}{175} \approx 0.023
\end{aligned}$$

We can use the same approach to determine the probability that a citizen has Covid given that their test is positive:

$$\begin{aligned}
\mathbb{P}[C|N^{\mathsf{c}}] &= \frac{\mathbb{P}[N^{\mathsf{c}}|C]\,\mathbb{P}[C]}{\mathbb{P}[N^{\mathsf{c}}|C]\,\mathbb{P}[C] + \mathbb{P}[N^{\mathsf{c}}|C^{\mathsf{c}}]\,\mathbb{P}[C^{\mathsf{c}}]} \\
&= \frac{0.8 \cdot 0.1}{0.8 \cdot 0.1 + 0.05 \cdot 0.9} \\
&= \frac{16}{25} = 0.64
\end{aligned}$$