# Exam 2 Solutions

**Problem 1** *16 points*

Let $X$ be a continuous random variable with the PDF $f_X(x) = \frac{1}{2}e^{-|x+1|}$ called the double exponential or Laplace distribution. The range of $X$ is $\mathbb{R}$, i.e., the set of all real numbers.

(a) (4 pts) Compute $\mathbb{E}[X]$. Your answer can be an integral, but you can also exploit symmetry to get an exact expression.

**Solution:** $\boxed{\mathbb{E}[X] = -1.}$

*Method 1 using symmetry:* The PDF $f_X(x)$ is symmetric around $-1$: $f_X(-1+t) = f_X(-1-t)$ for all $t$. Thus, $\mathbb{E}[X] = -1$.

*Method 2 using integrals:* $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} \frac{xe^{-|x+1|}}{2}dx = -1$

(b) (4 pts) Let $A = \{X > 0\}$. Compute expressions of $\mathbb{P}[X \in A]$ and the conditional PDF $f_{X|A}(x)$ (as a case-by-case formula). The answers are simple expressions, but they can be left in terms of integrals.

**Solution:** $\boxed{\mathbb{P}[A] = \frac{1}{2e},\ f_{X|A}(x) = \text{ PDF of an Exponential(1) RV.}}$

$$\mathbb{P}[A] = \int_0^{\infty} f_X(x)dx = \int_0^{\infty} \tfrac{1}{2}e^{-|x+1|}dx = \int_0^{\infty} \tfrac{1}{2}e^{-(x+1)}dx = \tfrac{e^{-1}}{2}\int_0^{\infty} e^{-x}dx = \tfrac{e^{-1}}{2} = \tfrac{1}{2e}$$

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}[X\in A]} & x \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} ee^{-|x+1|} & x > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} ee^{-(x+1)} & x > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Observe that the conditional PDF of $X$ when given that $X > 0$, is that of an Exponential(1) RV.

(c) (4 pts) Compute $\mathbb{P}[X < 2|X > 0]$. The answer is a simple expression, but it can be left in terms of integrals.

**Solution:** $\boxed{1 - e^{-2}}$

*Method 1 using conditional probability definition:*

$$\mathbb{P}[X < 2|X > 0] = \tfrac{\mathbb{P}[0<X<2]}{\mathbb{P}[X>0]} = \tfrac{\int_0^2 0.5e^{-|x+1|}dx}{\int_0^{\infty} 0.5e^{-|x+1|}dx} = \tfrac{\int_0^2 0.5e^{-(x+1)}dx}{\int_0^{\infty} 0.5e^{-(x+1)}dx} = \tfrac{\int_0^2 e^{-x}dx}{\int_0^{\infty} e^{-x}dx} = \tfrac{1-e^{-2}}{1-0} = 1 - e^{-2}$$

*Method 2 using conditional PDF from part (b)*

$$\mathbb{P}[X < 2|X > 0] = \mathbb{P}[X < 2|X \in A] = \int_{-\infty}^{2} f_{X|A}(x)dx = \int_{0}^{2} e^{-x}dx = 1 - e^{-2}$$

*Method 3 using properties of Exponential RVs:* Since the PDF of $X$ given $X > 0$ is that of an Exponential(1) RV, $\mathbb{P}[X < 2|X > 0] = 1 - e^{-2}$.

(d) (4 pts) Compute $\mathbb{E}[X|X > 0]$. The answer is a simple expression, but it can be left in terms of integrals.

**Solution:** $\boxed{1}$

*Method 1 using conditional expectation definition:*

$$\mathbb{E}[X|X > 0] = \mathbb{E}[X|X \in A] = \int_{-\infty}^{\infty} x f_{X|A}(x)dx = \int_{0}^{\infty} xe^{-x}dx = 1$$

*Method 2 using properties of Exponential RVs:* Since the PDF of $X$ given $X > 0$ is that of an Exponential(1) RV, $\mathbb{E}[X|X > 0] = 1$.

**Problem 2** *16 points*

Consider the pair of discrete random variables $X, Y$ with joint PMF described below:

| | $P_{X,Y}(x,y)$ | $y = -1$ | $y = 0$ | $y = 1$ | $y = 2$ |
|---|---|---|---|---|---|
| $x$ | 1 | 1/12 | 1/12 | 1/12 | 1/12 |
| | 2 | 0 | 1/6 | 1/6 | 0 |
| | 3 | 1/6 | 0 | 0 | 1/6 |

(a) (4 pts) Compute $\mathbb{P}[XY > 1]$.

**Solution:** $\boxed{\mathbb{P}[XY > 1] = 5/12}$

From the table, there are three non-zero entries with $XY > 1$: they are (1,2), (2,1) and (3,2). Their probabilities are 1/12, 1/6, and 1/6, so $\mathbb{P}[XY > 1] = 5/12$.

(b) (4 pts) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

**Solution:** $\boxed{\mathbb{E}[X] = 2, \mathbb{E}[Y] = 0.5}$

First, compute the marginal PMFs $P_X(x)$ and $P_Y(y)$ by summing along the rows and columns, respectively. This shows that

$P_X(1) = P_X(2) = P_X(3) = 1/3 \Rightarrow X \sim \text{Uniform}(1,3) \Rightarrow \mathbb{E}[X] = (1+3)/2 = 2$.

$P_Y(-1) = P_Y(0) = P_Y(1) = P_Y(2) = 1/4 \Rightarrow Y \sim \text{Uniform}(-1,2) \Rightarrow \mathbb{E}[Y] = (-1+2)/2 = 0.5$.

(c) (4 pts) Compute $\mathsf{Var}[Y|X = 3]$.

**Solution:** $\boxed{\mathsf{Var}[Y|X=3]=2.25}$

When $X=3$, $Y\sim 3\,\text{Bernoulli}(1/2)-1$. Since the variance of Bernoulli(1/2) is 1/4, $\mathsf{Var}[Y|X=3]=(3)^2*(1/4)=9/4=2.25$.

(d) (4 pts) Compute $\rho_{X,Y}$.

**Solution:** $\boxed{\rho_{X,Y}=0}$

*Method 1:* Using $\mathsf{Cov}[X,Y]=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]$, we need $\mathbb{E}[XY]$. Using the full joint PMF,

$$\mathbb{E}[XY]=\frac{1(1)(-1)+1(1)(1)+1(1)(2)+2(2)(1)+2(3)(-1)+2(3)(2)}{12}=1$$

Hence $\mathsf{Cov}[X,Y]=1-(2)(0.5)=0$ and $\rho_{X,Y}=0$ (you don't need to know the variances of $X$ and $Y$).

*Method 2:* Observe that each value of $y$, $P_{X,Y}[0.5-t,y]=P_{X,Y}[0.5+t,y]$, i.e., the joint PMF is symmetric with respect to the vertical line $x=0.5$. Therefore $X$ and $Y$ are uncorrelated and $\mathsf{Cov}[X,y]=\rho_{X,Y}=0$.

**Problem 3** *16 points*

You win a hundred dollars in the lottery! Feeling generous, you first give an amount $X\sim\text{Uniform}(0,100)$ of your winnings to one of your friends, and then give an amount $Y\sim\text{Uniform}(0,100\text{–}X)$ to another friend. Both $X$ and $Y$ are continuous random variables.

(a) (4 pts) Are $X$ and $Y$ independent? Why or why not?

**Solution:** No. Because the range of $Y$ given $X=x$ is $100-x$ which depends on the value of $x$.

(b) (4 pts) Compute the joint PDF $f_{X,Y}(x,y)$ and clearly state its range.

**Solution:**

$$f_X(x)=\begin{cases}\frac{1}{100} & 0\le x\le 100\\ 0 & \text{otherwise}\end{cases},\quad f_{Y|X}(y|x)=\begin{cases}\frac{1}{100-x} & 0\le y\le 100-x\\ 0 & \text{otherwise}\end{cases}$$

$$\Rightarrow f_{X,Y}(x,y)=f_X(x)f_{Y|X}(y|x)=\begin{cases}\frac{1}{100(100-x)} & 0\le x\le 100, 0\le y\le 100-x\\ 0 & \text{otherwise}\end{cases}$$

Range $R_{X,Y}=\{(x,y):0\le x\le 100, 0\le y\le 100-x\}$.

(c) (4 pts) Compute $\mathbb{E}[Y|X=20]$. For full credit, you must provide an exact numerical value.

**Solution:** $\boxed{\mathbb{E}[Y|X=20] = 40}$

$Y|X = 20 \sim \text{Uniform}(0, 100-20) \Rightarrow \mathbb{E}[Y|X=20] = \frac{0+(100-20)}{2} = 40\,.$

(d) (4 pts) Compute $\mathbb{E}[Y]$. For full credit, you must provide an exact numerical value.

**Solution:** $\boxed{\mathbb{E}[Y] = 25}$

$\mathbb{E}[X] = \frac{0+100}{2} = 50\,.$

$Y|X = x \sim \text{Uniform}(0, 100-x) \Rightarrow \mathbb{E}[Y|X] = \frac{0+(100-X)}{2} = 50 - \frac{X}{2}\,.$

By the law of total probability, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[50 - 0.5X] = 50 - 0.5\mathbb{E}[X] = 50 - 25 = 25\,.$

**Problem 4** *16 points*

Let $X_1$ and $X_2$ be independent standard Gaussian (zero mean, unit variance) RVs and $\underbrace{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}_{\underline{Y}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\underline{X}}$.

(a) (4 pts) Compute exact numerical values of the $2 \times 1$ mean vector $\mu_{\underline{Y}} = \mathbb{E}[\underline{Y}]$ and the $2 \times 2$ covariance matrix $\Sigma_{\underline{Y}} = \mathsf{Cov}[\underline{Y}]$.

**Solution:** $\boxed{\mu_{\underline{Y}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_{\underline{Y}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$

Since $X_1, X_2$ are independent standard Gaussian RVs,

$$\mu_{\underline{X}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underline{0}, \text{ the 2 x 1 zero vector and } \Sigma_{\underline{X}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \text{ the 2 x 2 identity matrix}$$

Therefore,

$$\mu_{\underline{Y}} = A\,\mu_{\underline{X}} = A\,\underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \Sigma_{\underline{Y}} = A\Sigma_{\underline{X}}A^\top = AI_2A^\top = AA^\top = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) (4 pts) Compute $\mathbb{E}[Y_1|X_1]$ as a function of the RV $X_1$.

**Solution:** $\boxed{\mathbb{E}[Y_1|X_1] = X_1}$

*Method 1:* Since $X_1, X_2$ are independent standard Gaussian RVs, $\mathbb{E}[Y_1|X_1] = \mathbb{E}[X_1 + X_2|X_1] = \mathbb{E}[X_1|X_1] + \mathbb{E}[X_2|X_1] = X_1 + \mathbb{E}[X_2] = X_1 + 0 = X_1\,.$

*Method 2:* First note that since $X_1, X_2$ are independent standard Gaussian RVs, they are uncorrelated and have unit variance, i.e., $\mathsf{Cov}[X_1, X_2] = 0$ and $\mathsf{Var}[X_1] = \mathsf{Cov}[X_1, X_1] = 1 = \mathsf{Var}[X_2]$.

Since $Y_1 = X_1 + X_2$ and $X_1 = X_1 + 0 \times X_2$ are linear functions of independent standard Gaussian RVs $X_1, X_2$, it follows that $X_1, Y_1$ are jointly Gaussian and therefore $Y_1|X_1 = x_1$ is a Gaussian RV with mean given by

$\mathbb{E}[Y_1|X_1=x_1]=\mu_{Y_1}+\frac{\mathsf{Cov}[Y_1,X_1]}{\mathsf{Var}[Y_1]}(x_1-\mu_{X_1})=0+\frac{\mathsf{Cov}[X_1+X_2,X_1]}{2}(x_1-0)=\frac{\mathsf{Cov}[X_1,X_1]+\mathsf{Cov}[X_2,X_1]}{2}x_1=\frac{x_1}{2}$.
Therefore, $\mathbb{E}[Y_1|X_1]=X_1/2$.

(c) (4 pts) Compute $\mathbb{P}[Y_1\le b|X_1=a]$ in terms of $a$, $b$ and the standard Gaussian CDF $\Phi(\cdot)$.

**Solution:** $\boxed{\mathbb{P}[Y_1\le b|X_1=a]=\Phi(b-0.5a)}$

From solution method 2 of part (b), $Y_1|X_1=a$ is a Gaussian RV with mean $E[Y_1|X_1=a]=0.5a$. We also have

$$\mathsf{Var}[Y_1|X_1=a]=\mathsf{Var}[Y_1]-\frac{(\mathsf{Cov}[Y_1,X_1])^2}{\mathsf{Var}[X_1]}=2-\frac{1^2}{1}=1.$$

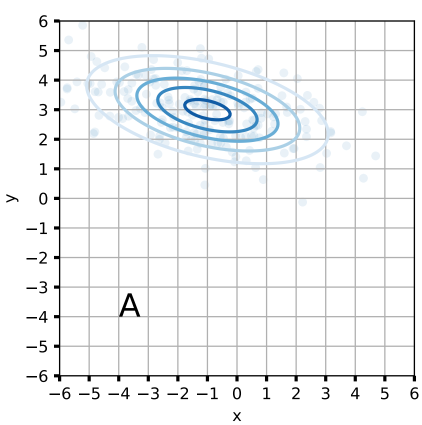
Therefore $\mathbb{P}[Y_1\le b|X_1=a]=\Phi\left(\frac{b-\mathbb{E}[Y_1|X_1=a]}{\sqrt{\mathsf{Var}[Y_1|X_1=a]}}\right)=\Phi\left(\frac{b-0.5a}{\sqrt{1}}\right)=\Phi(b-0.5a)$.

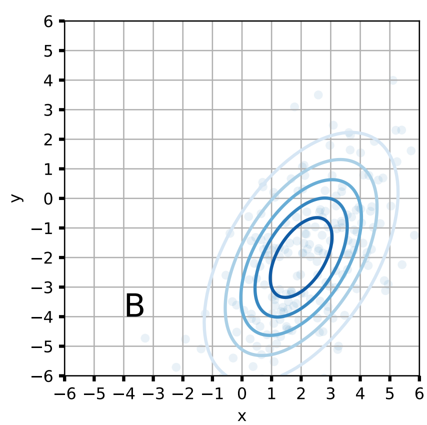
(d) (4 pts) Compute the exact numerical value of $\mathbb{E}[Y_1^2Y_2^2]$.

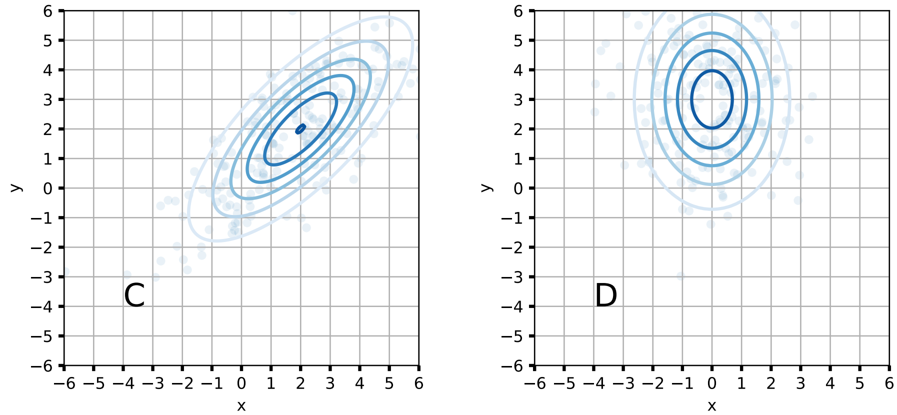
**Solution:** $\boxed{\mathbb{E}[Y_1^2Y_2^2]=4}$ Since $Y_1,Y_2$ are linear functions of independent standard Gaussian RVs $X_1,X_2$, they are jointly Gaussian. From part (a), $\mathsf{Cov}[Y_1,Y_2]=0\Rightarrow Y_1,Y_2$ are uncorrelated and since they are jointly Gaussian, they are independent RVs. Moreover, from part (a) $\mathbb{E}[Y_1]=\mathbb{E}[Y_2]=0$ and therefore $\mathbb{E}[Y_1^2]=\mathsf{Var}[Y_1]=2=\mathsf{Var}[Y_2]=\mathbb{E}[Y_2^2]$. Therefore, $\mathbb{E}[Y_1^2Y_2^2]=\mathbb{E}[Y_1^2]\,\mathbb{E}[Y_2^2]=2\times 2=4$.

**Problem 5** *16 points*

The table below depicts four jointly Gaussian PDFs via contour plots. In each case, the expectations, variances, and covariances are small integer values between $-4$ and $4$. Put a checkmark in the boxes in each column that you think are true for that contour plot. No justifications are needed and there may be multiple boxes checked per row and/or column.





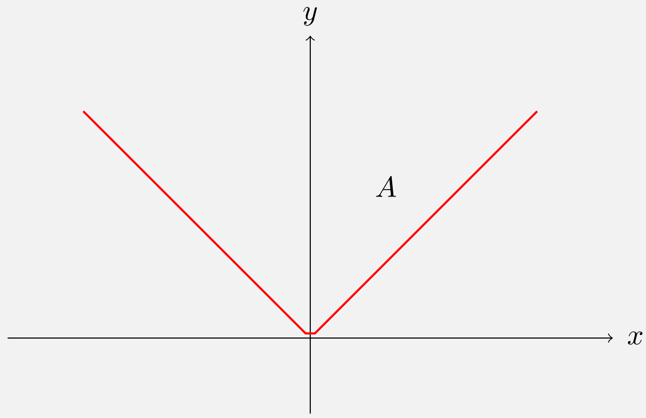

| | $\mathbb{E}[Y] > \mathbb{E}[X]$ | $\mathsf{Var}[Y] > \mathsf{Var}[X]$ | $\mathbb{P}[\, Y > \lvert X \rvert \,] > \frac{1}{2}$ | $\rho_{X,Y} > 0$ |
|---|---|---|---|---|
| A | ☐ | ☐ | ☐ | ☐ |
| B | ☐ | ☐ | ☐ | ☐ |
| C | ☐ | ☐ | ☐ | ☐ |
| D | ☐ | ☐ | ☐ | ☐ |

**Solution:**

The actual distributions are all jointly Gaussian, with the following parameters:

| | $\mathbb{E}[X]$ | $\mathbb{E}[Y]$ | $\mathsf{Var}[X]$ | $\mathsf{Var}[Y]$ | $\mathsf{Cov}[X,Y]$ |
|---|---|---|---|---|---|
| A | -1 | 3 | 5 | 1 | -1 |
| B | 2 | -2 | 3 | 5 | 2 |
| C | 2 | 2 | 4 | 4 | 3 |
| D | 0 | 3 | 2 | 4 | 0 |

Note that the event $\{Y > |X|\} = \{X \geq 0, Y > X\} \cup \{X < 0, Y > -X\} = \{(X,Y) \in A\}$ where $A$ is the V-shaped triangular region above the red curve $y = |x|$ shown in the figure below.




| | $\mathbb{E}[Y] > \mathbb{E}[X]$ | $\mathsf{Var}[Y] > \mathsf{Var}[X]$ | $\mathbb{P}[\, Y > \lvert X\rvert\,] > \frac{1}{2}$ | $\rho_{X,Y} > 0$ |
|---|---|---|---|---|
| A | ☑ | ☐ | ☑ | ☐ |
| B | ☐ | ☑ | ☐ | ☑ |
| C | ☐ | ☐ | ☐ | ☑ |
| D | ☑ | ☑ | ☑ | ☐ |

**Problem 6** *8 points*

Complete the following quick calculations. For full credit, you must work out a simplified, numerical answer for each requested quantity in this problem. The solutions do not require integration.

(a) (2pts) Let $X$ be a standard Gaussian RV. Compute $\mathbb{P}[\,|X| > 1|X < 2\,]$ in terms of the standard Gaussian CDF $\Phi(\cdot)$.

**Solution:** $\boxed{\mathbb{P}[\,|X| > 1|X < 2\,] = \frac{\phi(-1)+\phi(2)-\phi(1)}{\phi(2)}}$

$$\begin{aligned}\mathbb{P}[\,|X| > 1|X < 2\,] &= \frac{\mathbb{P}[\,\{|X| > 1\} \cap \{X < 2\}\,]}{\mathbb{P}[\,X < 2\,]} = \frac{\mathbb{P}[\,\{X < -1\}\,] + \mathbb{P}[\,\{X < 2\}\,] - \mathbb{P}[\,\{X < 1\}\,]}{\mathbb{P}[\,X < 2\,]} \\ &= \frac{\phi(-1) + \phi(2) - \phi(1)}{\phi(2)}\end{aligned}$$

(b) (2pts) Let $X$ be a continuous Uniform$(-1,1)$ RV. Compute $\mathbb{E}[X|X^2 > 0.25]$.

**Solution:** $\boxed{\mathbb{E}[X|X^2 > 0.25] = 0}$

The PDF of $X$ conditioned on the event $A = \{X^2 > 0.25\} = \{-1 \le X < -0.5\} \cup \{0.5 < X \le 1\}$, is uniform over $[-1,-0.5) \cup (0.5, 1]$ which is symmetric about $0$. Thus, $\mathbb{E}[X|X^2 > 0.25] = 0$.

(c) (2pts) Let $X$ be continuous Uniform$(-1,1)$ RV and let RV $Y$ given $X = x$ be Exponential$\left(\frac{1}{1+x^2}\right)$. Compute $\mathbb{E}[Y]$.

**Solution:** $\boxed{\mathbb{E}[Y] = 4/3 \approx 1.33}$

First note that since $X$ is a continuous Uniform$(-1,1)$ RV, $\mathbb{E}[X] = \frac{-1+1}{2} = 0$ and therefore $\mathbb{E}[X^2] = \mathsf{Var}[X] = \frac{(1-(-1))^2}{12} = \frac{4}{12} = \frac{1}{3}$.

Next, since $Y$ given $X = x$ is an Exponential$\left(\frac{1}{1+x^2}\right)$ RV, $\mathbb{E}[Y|X = x] = \frac{1}{\frac{1}{1+x^2}} = 1 + x^2$. Thus, $\mathbb{E}[Y|X] = 1 + X^2$.

Finally, by the law of total expectation, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[1 + X^2] = 1 + \frac{1}{3} = 4/3 \approx 1.33$.

(d) (2pts) Compute $\mathbb{E}[(X+Y)^2]$ if $X$ is Exponential$(1)$, $Y$ a standard Gaussian, and $\rho_{X,Y} = -0.5$.

**Solution:** $\boxed{\mathbb{E}[(X+Y)^2] = 2}$

Since $X$ be Exponential$(1)$, we have $\mu_X = \frac{1}{1} = 1$ and $\mathsf{Var}[X] = \frac{1}{1^2} = 1 \Rightarrow \mathbb{E}[X^2] = \mathsf{Var}[X] + \mu_X^2 = 2$.

Since $Y$ a standard Gaussian, we have $\mu_Y = 0$ and $\mathsf{Var}[Y] = 1 = \mathbb{E}[Y^2]$.

We also have $\mathsf{Cov}[X,Y] = \rho_{X,Y}\sqrt{\mathsf{Var}[X]}\sqrt{\mathsf{Var}[Y]} = -0.5 \times 1 \times 1 = -0.5$. Therefore, $\mathbb{E}[XY] = \mathsf{Cov}[X,Y] + \mu_X\mu_Y = \mathsf{Cov}[X,Y] = -0.5$

Finally, $\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] = 2 + 1 + 2(-0.5) = 2 + 1 - 1 = 2$.

**Problem 7** *8 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing T (for True) or F (for False) in **the box next to the question.** Full credit will be given for selecting the correct logical value (even with no explanation). Briefly explain your reasoning in the space provided for partial credit. Diagrams are welcome.

(a) (2pts) ☐ If $X$ is a continuous uniform RV with $\mathbb{E}[X] = 1$ and $\mathsf{Var}[X] = \frac{1}{3}$, then $\mathbb{P}[X < 0] = 0$.

**Solution:** $\boxed{\text{True}}$ Let $X$ be Uniform$(a,b)$. Then $(a+b)/2 = 1 \Rightarrow (a+b) = 2$ and $(b-a)^2/12 = 1/3 \Rightarrow (b-a) = 2$. This implies that $a = 0$ and $b = 2$ and the range of $X$ is $[0,2]$. Thus, $\mathbb{P}[X < 0] = 0$.

(b) (2pts) ☐ If $X$ is a continuous Uniform$(0,3)$ RV, then $Y = X^2$ is a continuous Uniform$(0,9)$ RV.

**Solution:** $\boxed{\text{False}}$ $\mathbb{P}[Y \leq 1] = \mathbb{P}[X \leq 1] = 1/3$. If $Y$ was a continuous Uniform$(0, 9)$ RV, then $\mathbb{P}[Y \leq 1] = 1/9$.

(c) (2pts) ☐ If $\mathsf{Var}[X] = 0.01$ and $\mathsf{Var}[Y] = 0.04$ then $\mathsf{Cov}[X, Y]$ can be $0.03$.

**Solution:** $\boxed{\text{False}}$ For the given values, $\rho_{X,Y} = \frac{\mathsf{Cov}[X,Y]}{\sqrt{\mathsf{Var}[X]\mathsf{Var}[Y]}} = \frac{0.03}{\sqrt{0.01 \times 0.04}} = \frac{0.03}{0.1 \times 0.2} = 1.5$ which is impossible, since $|\rho_{X,Y}| \leq 1$.

(d) (2pts) ☐ If $\mathsf{Var}[-2X + 3Y] = 4\mathsf{Var}[X] + 9\mathsf{Var}[Y]$, then $\mathsf{Var}[X - 2Y] = \mathsf{Var}[X] + 4\mathsf{Var}[Y]$.

**Solution:** $\boxed{\text{True}}$ $\mathsf{Var}[aX + bY] = a^2\mathsf{Var}[X] + b^2\mathsf{Var}[Y] + 2ab\mathsf{Cov}[X, Y]$. So if $\mathsf{Var}[-2X + 3Y] = 4\mathsf{Var}[X] + 9\mathsf{Var}[Y]$ then $\mathsf{Cov}[X, Y] = 0$ and therefore $\mathsf{Var}[X - 2Y] = \mathsf{Var}[X] + 4\mathsf{Var}[Y]$.

**Problem 8** *4 points*

$X$ and $Y$ are RVs with means $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, second moments $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$, and $\mathbb{E}[XY] = 0.5$. We want to estimate $Y$ using a linear function of $X$ given by

$$\widehat{Y} = uX + v$$

where $u$ and $v$ are constants to be designed. Compute the values of $u$ and $v$ which would make the mean squared error given by

$$g(u, v) = \mathbb{E}[(\widehat{Y} - Y)^2].$$

as small as possible, i.e., we want to minimize $g(u, v)$ with respect to variables $u$ and $v$.
*Useful algebraic identity:* $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

**Solution:** $\boxed{u = 0.5, v = 0}$

$$\begin{aligned} g(u, v) &= \mathbb{E}[(uX + v - Y)^2] \\ &= u^2\mathbb{E}[X^2] + v^2 + \mathbb{E}[Y^2] + 2uv\mathbb{E}[X] - 2v\mathbb{E}[Y] - 2u\mathbb{E}[XY] \\ &= u^2 + v^2 + 1 - u \\ &= (u - 0.5)^2 + v^2 + 0.75 \end{aligned}$$

Thus, $g(u, v)$ is minimized if $u = 0.5$ and $v = 0$ and the minimum mean squared error is $0.75$.