Exam 2

Last Name	First Name	Student ID #
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ______________________________

Partial Credit: The most important issue is knowing how to approach a particular problem. Therefore, there will be partial credit for good solution outlines even if not all the mathematical manipulations are completed correctly. Be sure to attempt every problem!

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed** - including no phones and no calculators.
- You can use the provided formula sheet handout - no extra materials are allowed.
- No form of collaboration is allowed.
- All work to be graded must be included in this document. Submit no extra sheets.
- There are 6 problems in total, with different point values listed below.
- If you need extra space, you can use the last blank page in the exam. Indicate that your answer continues in the main problem space so the grader can find it...

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		16	Problem 4		16
Problem 2		20	Problem 5		16
Problem 3		16	Problem 6		16
			Total		100

Problem 1 *16 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) Assume that X is a Gaussian random variable with mean 0 and variance 1. Then, the random variable $Y = X^3$ has $E[Y] = 0$.

Solution:

True. This is because of odd symmetry: the integral of an odd function is zero. The integral you want is $\int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2}}\, dx$ which is an odd function.

(b) Suppose $\mathsf{Cov}[X, Y] = 0$ and let $Z = 2X - Y$. Then, $\mathrm{Var}[Z] = 4\mathrm{Var}[X] - \mathrm{Var}[Y]$.

Solution:

False. Suppose $\mathsf{Var}[X] = 0$ but $\mathsf{Var}[Y] = 1$. We now get a negative $\mathsf{Var}[Z]$, which should be possible. The correct formula is $\mathsf{Var}[Z] = 4\mathsf{Var}[X] + \mathsf{Var}[Y]$.

(c) If $\mathbb{E}[X|Y = y] = Y^2$, then $\mathbb{E}[X] = \mathsf{Var}[Y] + \big(\mathbb{E}[Y]\big)^2$.

Solution:

True. By the iterated expectation property, we have that $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[Y^2] = \mathsf{Var}[Y] + \big(\mathbb{E}[Y]\big)^2$.

(d) If X, Y are jointly Gaussian and $Var(X + Y) = \mathsf{Var}[X] + \mathsf{Var}[Y]$, then X, Y are independent.

Solution:

True. $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) = Var(X) + Var(Y)$. Hence, $Cov(X, Y) = 0$, so X, Y are uncorrelated. Uncorrelated Gaussian random variables are independent.

(e) If X and Y are uncorrelated (meaning that $\mathsf{Cov}[X, Y] = 0$), then $\mathbb{E}[aX + b|Y] = a\mathbb{E}[X] + b$.

Solution:

False. $E[aX + b|Y] = aE[X|Y] + b$. The fact that X and Y are uncorrelated does not tell us that $E[X|Y] = E[X]$.

Problem 1 cont.

(f) If the range of X and Y is a rectangle, then they must be independent.

Solution:

False. Counterexample: $f_{X,Y}(x,y) = \begin{cases} (x+y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

(g) If $\rho_{X,Y} = 0$, then $\mathbb{E}[XY] = 0$.

Solution:

False. One can have $\mathbb{E}[X] = 1, \mathbb{E}[Y] = 1, \mathsf{Cov}[X,Y] = 0$ and $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 1$.

(h) Let $\underline{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ is a Gaussian random vector with mean $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and covariance $\Sigma_{\underline{Z}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
If $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, then X_1 and X_2 are independent.

Solution:

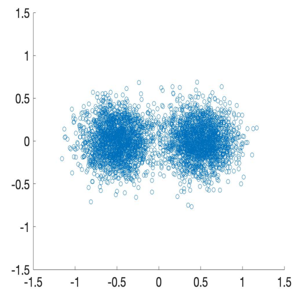
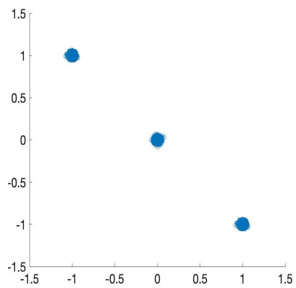
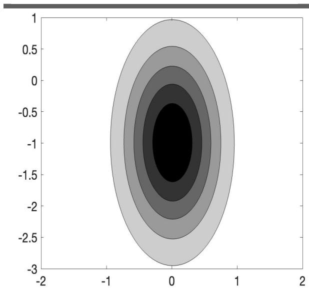
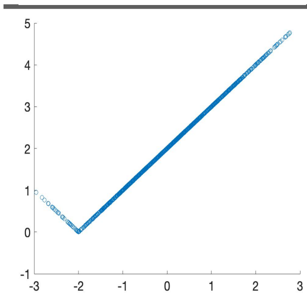
True. Since both X_1, X_2 are zero-mean, then

$$\mathsf{Cov}[X_1, X_2] = \mathbb{E}[(2Z_1 + Z_2)(\frac{-1}{2} Z_1 + Z_2)] = -\mathsf{Var}[Z_1] + \mathsf{Var}[Z_2] = 0$$

Since they are Gaussian and uncorrelated, they are independent.

Problem 2 *20 points*

The table below lists four scenarios via contour and scatter plots as well as equations.
Put a checkmark in the boxes in each row that you think are true for that scenario.
No justifications are needed and there may be multiple boxes checked per row and/or column.

	$\mathbb{E}[Y]$ noticeably larger than $\mathbb{E}[X]$	$\mathsf{Var}[Y]$ noticeably larger than $\mathsf{Var}[X]$	$\mathsf{Cov}[X, Y]$ close to 0	$\lvert\rho_{X,Y}\rvert$ close to 1
	☐	☐	✓	☐
	☐	☐	☐	✓
	☐	✓	✓	☐
	✓	☐	☐	✓
U, V jointly Gaussian $\mathbb{E}[U] = \mathbb{E}[V] = 0$ $\mathsf{Var}[U] = \mathsf{Var}[V] = 1$ $\mathsf{Cov}[U, V] = -0.5$ $X = U + 2V; Y = U - V + 1$	✓	☐	☐	☐

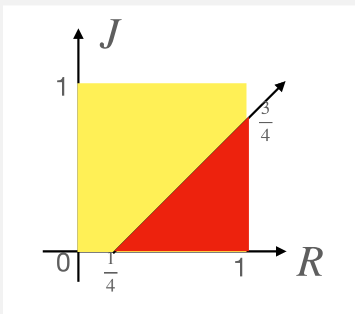
Problem 3 *16 points*

For each of the following parts, calculate the **two requested quantities exactly.** You should arrive at a numerical answer in each part. For this particular problem, you may **not** leave your answer in terms of integrals. Show your steps for partial credit

(a) Romeo and Juliet have a dinner date tonight. Each person will arrive at the restaurant with a delay uniformly distributed between zero and one hour, independent of the other person. Let $R \in [0,1]$ and $J \in [0,1]$ denote the delay of Romeo and Juliet, respectively. Sketch the range $R_{R,J}$ of the joint PDF $f_{R,J}(r,j)$, and compute the probability that Romeo arrives 1/4 of an hour or more later than Juliet (That is, compute $\mathbb{P}[R - J \geq 1/4]$.)

Solution:

The range is a square, with corners (0,0), (0,1), (1,0), (1,1). The range is illustrated in the figure below.



The area where Romeo arrives 1/4 of an hour or more later than Juliet is illustrated in red. We see that the probability is the area of the triangle, which is $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{32}$.

(b) Let X and Y be joint Gaussian random variables, with Y having mean 0, variance 1 ($f_Y(y) \sim \mathcal{N}(0,1)$), and $f_{X|Y}(x|y) \sim \mathcal{N}(y,1)$ is Gaussian with mean y and variance 1. Compute $\mathbb{E}[X]$ and $\mathsf{Var}[X]$.

Solution:

Using the law of total expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}\left[Y\right] = 0$$

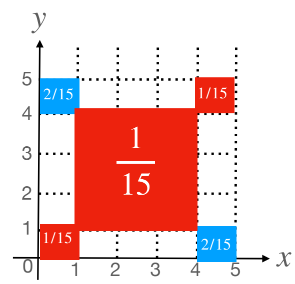
$$\mathbb{E}[X^2] = \mathbb{E}\left[\mathbb{E}[X^2|Y]\right] = \mathbb{E}\left[1 + Y^2\right] = 1 + 1 = 2$$

$$\mathsf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X^2])^2 = 2.$$

Problem 3 cont.

(c) The joint pdf of X, Y is defined below and illustrated in the figure, where the density is $1/15$ in the red areas and $2/15$ in the blue areas.

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{15} & x \in [0,1], y \in [4,5], \text{ or } y \in [0,1], x \in [4,5] \\ \frac{1}{15} & x \in [1,4], y \in [1,4] \\ \frac{1}{15} & x \in [0,1], y \in [0,1], \text{ or } x \in [4,5], y \in [4,5] \\ 0 & \text{elsewhere.} \end{cases}$$



Compute the marginal density $f_X(x)$ and the conditional density of X given $Y = 0.5$.

Solution:

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\, dy = \begin{cases} \frac{1}{5} & 0 \leq x \leq 5 \\ 0 & \text{elsewhere.} \end{cases}$$

$$f_Y(0.5) = \int_0^1 f_{X,Y}(x,y)\, dx = \frac{1}{5}$$

$$f_{X,Y}(x|0.5) = \frac{f_{X,Y}(x, 0.5)}{f_Y(0.5)} = \begin{cases} \frac{1}{3} & 0 \leq x \leq 1 \\ \frac{2}{3} & 4 \leq x \leq 5 \\ 0 & \text{elsewhere.} \end{cases}$$

(d) The joint PMF of two random variables X and Y are given in the table on the right. Determine the quantities $\mathbb{E}[Y]$ and $\mathsf{Cov}(X,Y)$.

$P_{XY}(x,y)$		y		
		-1	0	1
x	-1	$1/5$	0	$1/5$
	0	0	$1/5$	0
	1	$1/5$	0	$1/5$

Solution:

By symmetry, $\mathbb{E}[Y] = 0$.

$\mathsf{Cov}[X,Y] = \mathbb{E}[XY] = 0$ by symmetry also.

Problem 4 *16 points*

While exponential and Gaussian random variables have PDFs that decay exponentially as $x \to \infty$, there are other distributions of interest which decay only as the inverse of a polynomial in x. Consider a continuous random variable X with PDF

$$f_X(x) = \begin{cases} Cx^{-3.5} & 1 \le x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

A useful fact for this problem: $\int_b^\infty x^{-\alpha}\, dx = \frac{b^{1-\alpha}}{\alpha-1}$ if $\alpha > 1$.

(a) Calculate the value of C as a number using the given fact.

> **Solution:**
>
> By normalization, we have
>
> $$\int_1^\infty Cx^{-3.5}\, dx = \frac{C}{3.5-1} = 1$$
>
> which implies $C = 2.5$.

(b) Calculate $\mathbb{E}[X]$ as a number.

> **Solution:**
>
> $$\mathbb{E}[X] = \int_1^\infty x 2.5x^{-3.5}\, dx = \int_1^\infty 2.5x^{-2.5}\, dx = \frac{2.5}{1.5} = \frac{5}{3}$$

(c) Calculate $\mathsf{Var}[X]$ as a number.

> **Solution:**
>
> $$\mathbb{E}[X^2] = \int_1^\infty x^2 2.5x^{-3.5}\, dx = \int_1^\infty 2.5x^{-1.5}\, dx = \frac{2.5}{0.5} = 5$$
>
> $$\mathsf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 5 - \frac{25}{9} = \frac{20}{9}$$

(d) Let B be the event that $X > 2$. Compute $\mathbb{E}[X|B]$ as a number.

> **Solution:**
>
> $$\mathbb{P}[B] = \int_2^\infty 2.5x^{-3.5}\, dx = 2^{-2.5}$$
>
> $$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}[B]} & x \in B \\ 0 & \text{otherwise.} \end{cases}$$
>
> $$\mathbb{E}[X|X>2] = \frac{1}{\mathbb{P}[B]} \int_2^\infty x 2.5x^{-3.5}\, dx = 2^{2.5} \cdot \frac{2.5 \times 2^{-1.5}}{1.5} = \frac{10}{3}$$

Problem 5 *16 points*

Consider the joint probability mass function

$P_{XY}(x,y)$		$y=-1$	$y=0$	$y=1$	$y=2$
x	-1	0.04	0.10	0.04	0.02
	0	0.10	0.35	0.10	0.04
	1	0.04	0.10	0.04	0.03

(a) Calculate the probability $\mathbb{P}[\sqrt{X^2+Y^2} < 1]$.

Solution:

The only pair (x,y) for which $\sqrt{x^2+y^2} < 1$ is $(x,y) = (0,0)$. Therefore $\mathbb{P}[\sqrt{X^2+Y^2} < 1] = 0.35$.

(b) Calculate the marginal probability mass function $P_X(x)$.

Solution:

$$P_X(x) = \begin{cases} 0.04 + 0.10 + 0.04 + 0.02 = 0.20, & x = -1 \\ 0.10 + 0.35 + 0.10 + 0.04 = 0.59, & x = 0 \\ 0.04 + 0.10 + 0.04 + 0.03 = 0.31, & x = 1 \end{cases}$$

(c) Are X and Y independent?

Solution:

The probability that $X = -1$ is $P_X(-1) = 0.20$. The probability that $Y = -1$ is $P_Y(-1) = 0.04 + 0.10 + 0.04 = 0.18$. Since $P_{XY}(-1,-1) \neq P_X(-1)P_Y(-1)$, X and Y are not independent.

(d) Calculate the conditional probability $\mathbb{P}[X = 1 | X > Y]$.

Solution:

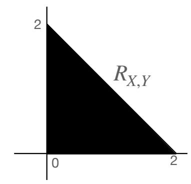
The pairs (x,y) for which $x > y$ are $(x,y) = (0,-1), (1,-1), (1,0)$. The pairs (x,y) for which $x = 1$ and $x > y$ are $(x,y) = (1,-1), (1,0)$. Therefore

$$\mathbb{P}[X = 1 | X > Y] = \frac{\mathbb{P}[X = 1 \text{ and } X > Y]}{\mathbb{P}[X > Y]} = \frac{0.04 + 0.10}{0.10 + 0.04 + 0.10} = \frac{0.14}{0.24} = \frac{7}{12}$$

Problem 6 *16 points*

Consider the pair of random variables X, Y with joint probability density function defined below, where the figure on the left illustrates the range $R_{X,Y}$:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}y & \text{if } 0 < x < 2, 0 < y < 2-x \\ 0 & \text{otherwise.} \end{cases}$$



Note that the density is not constant over its range.

(a) Calculate the marginal density $f_Y(y)$. Do not leave the answer as an integral.

Solution:

$f_Y(y) = \int_0^{2-y} \frac{3}{4}y\,dx = \frac{3}{4}y(2-y)$, for $y \in (0,2)$.

(b) Calculate $\mathbb{E}[X^2Y^2]$. You can leave the answer as an integral.

Solution:

$\mathbb{E}[X^2Y^2] = \int_0^2 \int_0^{2-x} \frac{3}{4}(x^2y^2)y\,dy\,dx$

(c) Calculate $\mathbb{E}[X|Y=y]$ for a given $y \in [0,2]$. This cannot be an integral.

Solution:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{3}{4}y}{\frac{3}{4}(y)(2-y)} = \frac{1}{2-y}$$

Given $Y = y$, X is uniformly distributed on $[0, 2-y]$. Thus, $\mathbb{E}[X|Y=y] = 1 - \frac{y}{2}$, for $y \in (0,2)$.

(d) Calculate $\mathsf{Var}[X|Y=y]$. This cannot be an integral.

Solution:

Using the idea from the previous part, $\mathsf{Var}[X|Y=y] = \frac{(2-y)^2}{12}$.