Exam 2

First Name	Last Name	UID
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ______________________________

Partial Credit: There will be partial credit for solution attempts even if not all the mathematical manipulations are completed correctly. To maximize your chances for partial credit, attempt every problem.

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed**, including no phones and no calculators. No form of collaboration is allowed.
- Unless explicitly stated otherwise, you may leave your answers in terms of integrals with the correct integrand expression and the correct integration limits. For PDFs, you must specify the correct range.
- You can use the provided formula sheet handouts – no other materials are allowed.
- All work to be graded must be included in this document. Submit no extra sheets.
- $\boxed{\text{Box}}$ your final answers.
- There are 8 problems in total, with different point values.

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		16	Problem 5		16
Problem 2		16	Problem 6		8
Problem 3		16	Problem 7		8
Problem 4		16	Problem 8		4
			Total		100

Problem 1 *16 points*

Let X be a continuous random variable with the PDF $f_X(x) = \frac{1}{2}e^{-|x+1|}$ called the double exponential or Laplace distribution. The range of X is $\mathbb{R}$, i.e., the set of all real numbers.

(a) (4 pts) Compute $\mathbb{E}[X]$. Your answer can be an integral, but you can also exploit symmetry to get an exact expression.

(b) (4 pts) Let $A = \{X > 0\}$. Compute expressions of $\mathbb{P}[X \in A]$ and the conditional PDF $f_{X|A}(x)$ (as a case-by-case formula). The answers are simple expressions, but they can be left in terms of integrals.

(c) (4 pts) Compute $\mathbb{P}[X < 2|X > 0]$. The answer is a simple expression, but it can be left in terms of integrals.

(d) (4 pts) Compute $\mathbb{E}[X|X > 0]$. The answer is a simple expression, but it can be left in terms of integrals.

Problem 2 *16 points*

Consider the pair of discrete random variables X, Y with joint PMF described below:

$P_{X,Y}(x,y)$		$y=-1$	$y=0$	$y=1$	$y=2$
x	1	1/12	1/12	1/12	1/12
	2	0	1/6	1/6	0
	3	1/6	0	0	1/6

(a) (4 pts) Compute $\mathbb{P}[XY > 1]$.

(b) (4 pts) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(c) (4 pts) Compute $\mathsf{Var}[Y|X = 3]$.

(d) (4 pts) Compute $\rho_{X,Y}$.

Problem 3 *16 points*

You win a hundred dollars in the lottery! Feeling generous, you first give an amount $X \sim \text{Uniform}(0, 100)$ of your winnings to one of your friends, and then give an amount $Y \sim \text{Uniform}(0, 100–X)$ to another friend. Both X and Y are continuous random variables.

(a) (4 pts) Are X and Y independent? Why or why not?

(b) (4 pts) Compute the joint PDF $f_{X,Y}(x, y)$ and clearly state its range.

(c) (4 pts) Compute $\mathbb{E}[Y|X = 20]$. For full credit, you must provide an exact numerical value.

(d) (4 pts) Compute $\mathbb{E}[Y]$. For full credit, you must provide an exact numerical value.

Problem 4 *16 points*

Let X_1 and X_2 be independent standard Gaussian (zero mean, unit variance) RVs and $\underbrace{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}_{\underline{Y}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\underline{X}}$.

(a) (4 pts) Compute exact numerical values of the 2×1 mean vector $\mu_{\underline{Y}} = \mathbb{E}[\underline{Y}]$ and the 2×2 covariance matrix $\Sigma_{\underline{Y}} = \mathsf{Cov}[\underline{Y}]$.

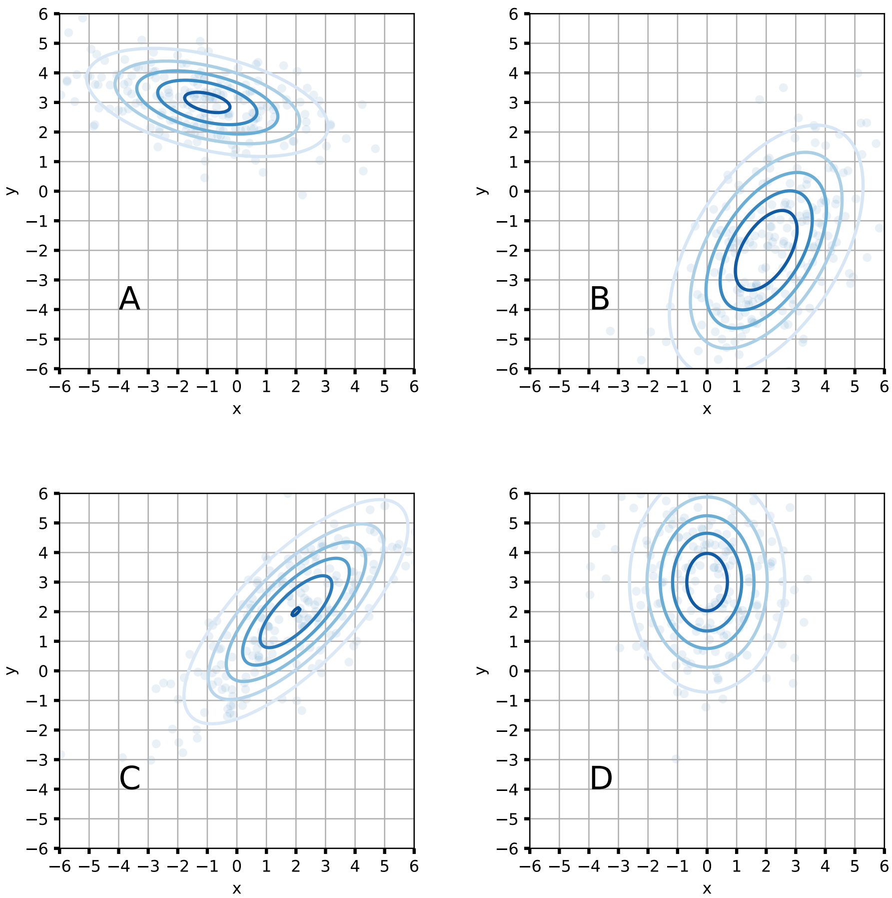
(b) (4 pts) Compute $\mathbb{E}[Y_1 | X_1]$ as a function of the RV X_1.

(c) (4 pts) Compute $\mathbb{P}[Y_1 \leq b | X_1 = a]$ in terms of a, b and the standard Gaussian CDF $\Phi(\cdot)$.

(d) (4 pts) Compute the exact numerical value of $\mathbb{E}[Y_1^2 Y_2^2]$.

Problem 5 *16 points*

The table below depicts four jointly Gaussian PDFs via contour plots. In each case, the expectations, variances, and covariances are small integer values between -4 and 4. Put a checkmark in the boxes in each column that you think are true for that contour plot. No justifications are needed and there may be multiple boxes checked per row and/or column.



	$\mathbb{E}[Y] > \mathbb{E}[X]$	$\mathsf{Var}[Y] > \mathsf{Var}[X]$	$\mathbb{P}[\, Y > \lvert X \rvert \,] > \frac{1}{2}$	$\rho_{X,Y} > 0$
A	☐	☐	☐	☐
B	☐	☐	☐	☐
C	☐	☐	☐	☐
D	☐	☐	☐	☐

Problem 6 *8 points*

Complete the following quick calculations. For full credit, you must work out a simplified, numerical answer for each requested quantity in this problem. The solutions do not require integration.

(a) (2pts) Let X be a standard Gaussian RV. Compute $\mathbb{P}[\,|X| > 1|X < 2\,]$ in terms of the standard Gaussian CDF $\Phi(\cdot)$.

(b) (2pts) Let X be a continuous Uniform$(-1, 1)$ RV. Compute $\mathbb{E}[X|X^2 > 0.25]$.

(c) (2pts) Let X be continuous Uniform$(-1, 1)$ RV and let RV Y given $X = x$ be Exponential$\left(\frac{1}{1+x^2}\right)$. Compute $\mathbb{E}[Y]$.

(d) (2pts) Compute $\mathbb{E}[(X + Y)^2]$ if X is Exponential(1), Y a standard Gaussian, and $\rho_{X,Y} = -0.5$.

Problem 7 *8 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing T (for True) or F (for False) in **the box next to the question.** Full credit will be given for selecting the correct logical value (even with no explanation). Briefly explain your reasoning in the space provided for partial credit. Diagrams are welcome.

(a) (2pts) $\square$ If X is a continuous uniform RV with $\mathbb{E}[X] = 1$ and $\mathsf{Var}[X] = \frac{1}{3}$, then $\mathbb{P}[X < 0] = 0$.

(b) (2pts) $\square$ If X is a continuous Uniform$(0, 3)$ RV, then $Y = X^2$ is a continuous Uniform$(0, 9)$ RV.

(c) (2pts) $\square$ If $\mathsf{Var}[X] = 0.01$ and $\mathsf{Var}[Y] = 0.04$ then $\mathsf{Cov}[X, Y]$ can be 0.03.

(d) (2pts) $\square$ If $\mathsf{Var}[-2X + 3Y] = 4\mathsf{Var}[X] + 9\mathsf{Var}[Y]$, then $\mathsf{Var}[X - 2Y] = \mathsf{Var}[X] + 4\mathsf{Var}[Y]$.

Problem 8 *4 points*

X and Y are RVs with means $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, second moments $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$, and $\mathbb{E}[XY] = 0.5$. We want to estimate Y using a linear function of X given by

$$\widehat{Y} = uX + v$$

where u and v are constants to be designed. Compute the values of u and v which would make the mean squared error given by

$$g(u, v) = \mathbb{E}[(\widehat{Y} - Y)^2].$$

as small as possible, i.e., we want to minimize $g(u, v)$ with respect to variables u and v.
Useful algebraic identity: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.