Exam 2

Last Name	First Name	Student ID #
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ______________________________

Partial Credit: There will be partial credit for solution attempts even if not all the mathematical manipulations are completed correctly. It is advisable to attempt every problem.

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed** – no phones and no calculators. No form of collaboration is allowed.
- You can use the provided formula sheet handout – no other materials are allowed.
- All work to be graded must be included in this document. Submit no extra sheets. Page 10 is blank and can be used for scratch work, but the final answer to each part must be written back in the page with the problem statement.
- Answers can be left in terms of sums without having to add or multiply all the terms, unless otherwise specified in the problem.
- There are 8 problems in total, with different point values. Don't get bogged down with any one true/false question. The last problem is **worth little, but is conceptually harder**; attempt all other problems before this one.

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		10	Problem 5		12
Problem 2		16	Problem 6		12
Problem 3		16	Problem 7		14
Problem 4		16	Problem 8		4
			Total		100

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing T (for True) or F (for False) in **the box next to the question.** Full credit will be given for selecting the correct logical value (even with no explanation). Briefly explain your reasoning in the space provided for partial credit. Diagrams are welcome.

(a) ☐ The function on the right is a valid Cumulative Distribution Function for a continuous random variable X:

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x/2 & 0 < x \leq 1 \\ \frac{1}{2} & 1 < x \leq 3 \\ \frac{x-2}{2} & 3 < x \leq 4 \\ 1 & x > 4 \end{cases}$$

(b) ☐ Let X, Y be a pair of uncorrelated random variables. Then, $\mathbb{E}[X|Y] = \mathbb{E}[X]$.

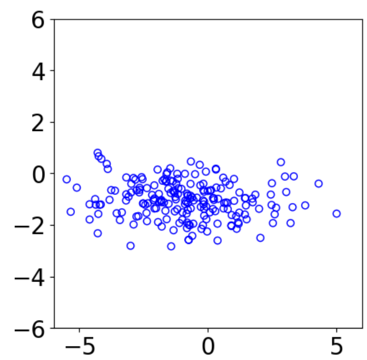
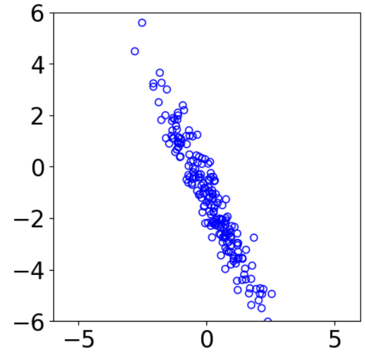
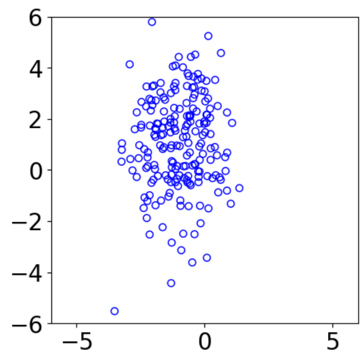
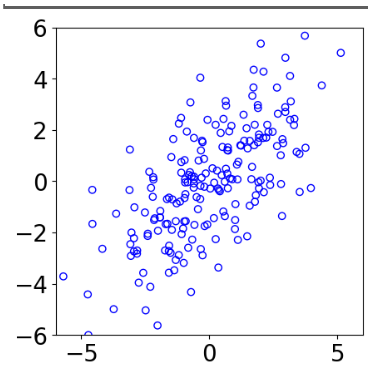
(c) ☐ Let X, Y be joint Gaussian random variables, uncorrelated, zero mean, each with variance 1. Let $U = X + 2Y$, $V = Y - 2X$. Then, U and V are independent.

(d) ☐ If X, Y are uncorrelated random variables, and $Z = 2X - Y$, then $\mathsf{Var}[Z] = 4\mathsf{Var}(X) - \mathsf{Var}[Y]$.

(e) ☐ Let X, Y be independent Exponential(1) random variables. Then, $Z = X+Y$ is an Exponential($\frac{1}{2}$) random variable.

Problem 2 *10 points*

The table below lists four joint PDFs illustrated by scatter plots of their samples.
Put a checkmark in the boxes in each row that you think are true for that joint PDF.
No justifications are needed and there may be multiple boxes checked per row and/or column.

	Cov$[X, Y]$ is positive	Var$[Y]$ noticeably larger than Var$[X]$	Cov$[X, Y]$ close to 0	$\lvert\rho_{X,Y}\rvert$ close to 1
	☐	☐	☐	☐
	☐	☐	☐	☐
	☐	☐	☐	☐
	☐	☐	☐	☐

Problem 3 Complete the following quick calculations. *16 points*

(a) Let X be a Gaussian random variable with $\mathbb{E}[X] = 1, \mathsf{Var}[X] = 4$. Compute $\mathbb{E}[(2X+1)^2]$ and $\mathbb{P}[X \geq 3]$. You can leave your answer in terms of the standard Gaussian CDF $\Phi(\cdot)$.

(b) Let $\underline{X}$ be a joint two-dimensional Gaussian vector with mean $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and covariance matrix $\mathbf{\Sigma}_{\underline{X}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Let $\underline{Y} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \underline{X}$. Compute the covariance matrix $\mathbf{\Sigma}_{\underline{Y}} \triangleq \mathsf{Cov}[\underline{Y}]$.

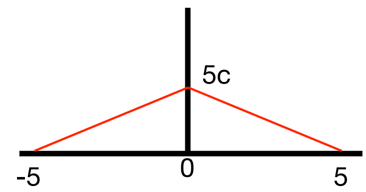
(c) Let X a discrete random variable uniformly distributed on $\{1, 2, 3\}$, so $\mathbb{P}[X = 1] = \mathbb{P}[X = 2] = \mathbb{P}[X = 3] = \frac{1}{3}$. Let Y given $X = x$ be Poisson($2x$) random variable. Compute $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$.

(d) Let X be an Exponential($\frac{1}{4}$) random variable. Compute $\mathbb{P}[X \geq 4]$ and $\mathbb{P}[X \geq 10 \mid X \geq 4]$.

Problem 4 *16 points*

Consider a continuous random variable X with the following PDF:

$$f_X(x) = \begin{cases} c\ (5-x) & 0 \le x \le 5 \\ c\ (5+x) & -5 \le x < 0 \\ 0 & |x| > 5 \end{cases}$$



The density is illustrated in the diagram on the right.

(a) Determine the value of c that satisfies the normalization property. Set c to this value for the remainder of the problem. This can be done without integrating, so this should be a number.

(b) What is the expected value of X? Your answer can be an integral, but you can also exploit symmetry.

(c) What is the variance of X? Your answer can be an integral.

(d) Calculate $\mathbb{P}[X > 1]$. Your answer can be an integral.

(e) Calculate $\mathbb{P}[X > 4 | X > 1]$. Your answer can be left in terms of integrals.

Problem 5 *12 points*

Let X be a Binomial $(2, \frac{1}{2})$ random variable.

Given that $X = x$, $P_{Y|X}(y|x) = \begin{cases} 1/3 & y = x; \\ 2/3 & y = x+1; \\ 0 & \text{otherwise} \end{cases}$

(a) (4 pts) Write down the joint Probability Mass Function (PMF) of X and Y as a table on the right.

$P_{XY}(x,y)$		$y=0$	$y=1$	$y=2$	$y=3$
x	0				
	1				
	2				

(b) (2 pts) Are X and Y independent? Explain why or why not.

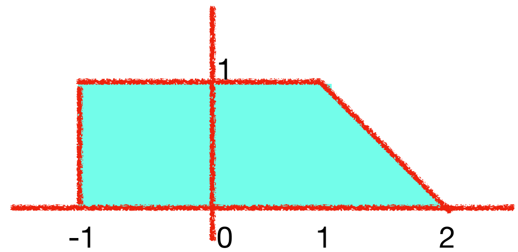
(c) (2 pts) Determine the marginal PMF of Y.

(d) (2 pts) What is the probability that $Y - X = 1$?

(e) (2 pts) Write down the conditional PMF of X given $Y = 1$.

Problem 6 *12 points*

Consider two jointly continuous random variables X, Y with joint PDF given below. The PDF is uniform in the range $R_{X,Y}$ illustrated in the figure on the right.



$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{5} & y \in [0,1], x \in [-1, 2-y], \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X, Y independent? Explain why or why not.

(b) Compute $\mathbb{E}[Y]$. Full credit will only be given for numerical answers, not integrals.

(c) Compute $\mathbb{E}[X|Y = 0.5]$. Full credit only for numerical answers, not integrals.

(d) Compute $\mathsf{Var}[X|Y = 0.5]$. Full credit only for numerical answers, not integrals.

Problem 7 *14 points*

Assume that X, Y are uncorrelated, jointly Gaussian random variables, such that $\mathbb{E}[X] = 0, \mathbb{E}[Y] = 1, \mathsf{Var}[X] = 1, \mathsf{Var}[Y] = 2$. Define $A = 4X + 2Y - 2, B = 2X + 4Y - 2$

(a) Compute $\mathbb{E}[A], \mathbb{E}[B]$.

(b) Compute $\mathsf{Var}[A], \mathsf{Var}[B]$.

(c) Compute $\mathsf{Cov}[A, B]$.

(d) Are A, B independent? Explain.

(e) Compute $\mathbb{E}[A|B]$

(f) Let $e = A - \mathbb{E}[A|B]$. Compute $\mathbb{E}[e^2]$, which is the conditional variance of A given B.

(g) Let $Z = X^2Y^2$ be a new random variable. Compute $\mathbb{E}[Z]$.

Problem 8 *4 points*

Let X, Y be independent Gaussian random variables with mean 0 and variance 1. Thus, the joint PDF of X, Y is

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

1. (1 pt) For a given number z, compute $\mathbb{P}[\{Y > 0, X \leq zY\}]$. You can leave the answer as an integral that depends on z.

2. (1 pt) For a given number z, compute $\mathbb{P}[\{Y < 0, X \geq zY\}]$. You can leave the answer as an integral that depends on z.

3. (2 pts) Observe the following: if we define the variable $Z = \frac{X}{Y}$, the CDF of Z can be computed as

$$F_Z(z) = \mathbb{P}[\frac{X}{Y} \leq z] = \mathbb{P}[\{Y > 0, X \leq zY\}] + \mathbb{P}[\{Y < 0, X \geq zY\}]$$

   Using the answer from the previous two parts, compute the PDF $f_Z(z)$.