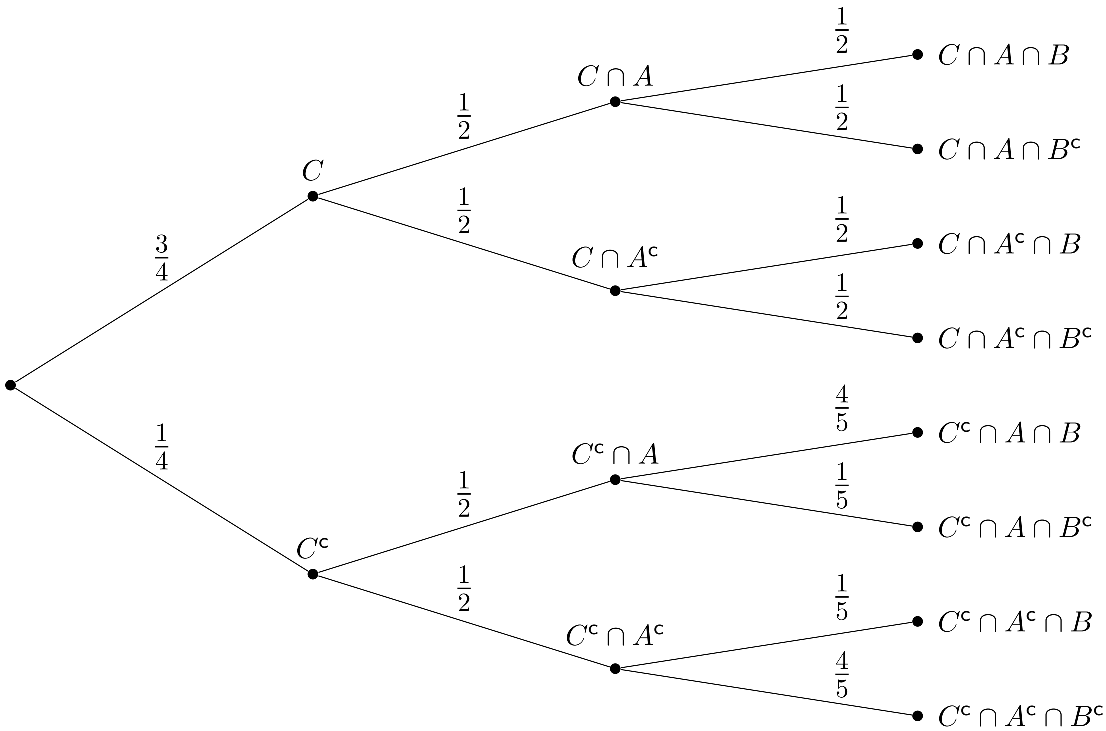
Exam 1 Solutions

Problem 1 *12 points*

The below probability tree describes the conditional probabilities involving three events A, B, C:



(a) Calculate $\mathbb{P}[A \cap C^c]$ and $\mathbb{P}[A \cap B \cap C^c]$ (4 pts)

Solution:

$\boxed{\text{Answers: } \mathbb{P}[A \cap C^c] = 1/8,\ \mathbb{P}[A \cap B \cap C^c] = 1/10}$ Explanation:

$$\mathbb{P}[A \cap C^c] = \mathbb{P}[C^c]\ \mathbb{P}[A|C^c] = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}, \quad \mathbb{P}[A \cap B \cap C^c] = \mathbb{P}[A \cap C^c]\ \mathbb{P}[B|A \cap C^c] = \frac{1}{8} \times \frac{4}{5} = \frac{1}{10},$$

(b) Calculate $\mathbb{P}[B \cap C^c]$ (4 pts)

Solution:

$\boxed{\text{Answer: } \mathbb{P}[B \cap C^c] = 1/8}$ Explanation:

$$\mathbb{P}[B \cap C^c] = \mathbb{P}[C^c \cap A \cap B] + \mathbb{P}[C^c \cap A^c \cap B] = \frac{1}{4} \times \frac{1}{2} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{2} \times \frac{1}{5} = \frac{1}{8}$$

(c) Determine if events A and B are conditionally independent given event C^c. Justify your answer with calculations. (4 pts)

Solution:

$\boxed{\text{Answer: } A \text{ and } B \text{ are not conditionally independent given even } C^c}$ Explanation:

$\mathbb{P}[A \cap B|C^c] = \mathbb{P}[A|C^c]\,\mathbb{P}[B|A \cap C^c] = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$.

$\mathbb{P}[A|C^c]\,\mathbb{P}[B|C^c] = \mathbb{P}[A|C^c]\,\frac{\mathbb{P}[B \cap C^c]}{\mathbb{P}[C^c]} = \frac{1}{2} \times \frac{1/8}{1/4} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Thus, $\mathbb{P}[A \cap B|C^c] \neq \mathbb{P}[A|C^c]\,\mathbb{P}[B|C^c]$ and therefore A and B are not conditionally independent given C^c.

Problem 2 *16 points*

You have ten coins: one penny, two nickels, three dimes, and four quarters. You randomly select one coin out of the ten, and then randomly select another coin out of the remaining nine. During each selection, all available coins are equally likely to be selected.

(a) Calculate the probability that both coins that you select are dimes. (4 pts)

Solution:

$\boxed{\text{Answer: } 1/15}$

Method 1 (keeping track of selection order): The number of ordered selections without replacement of 2 coins out of 10 coins is 10×9. Out of these, the number of selections in which both coins are dimes is 3×2 (since there are 3 dimes). Since during each selection, all available coins are equally likely to be selected, the required probability is:

$$\frac{3 \times 2}{10 \times 9} = \frac{1}{15}$$

Method 2 (ignoring selection order): The number of unordered selections without replacement of 2 coins out of 10 coins is $\binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45$. Out of these, the number of selections in which both coins are dimes is $\binom{3}{2} = 3$ (since there are 3 dimes). Since during each selection, all available coins are equally likely to be selected, the required probability is:

$$\frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45} = \frac{1}{15}$$

(b) Calculate the probability that at least one coin that you select is a dime. (4 pts)

Solution:

$\boxed{\text{Answer: } 8/15}$ $\mathbb{P}[\text{at least one dime}] = 1 - \mathbb{P}[\text{no dimes}] = 1 - \frac{\binom{7}{2}}{\binom{10}{2}} = 1 - \frac{7 \times 6}{10 \times 9} = \frac{48}{90} = \frac{8}{15}$.

(c) Calculate the probability that the second coin is a dime, given that the first coin is a nickel. (4 pts)

Solution:

$\boxed{\text{Answer: } 1/3}$

$\mathbb{P}[\text{2nd is dime}|\text{1st is nickel}] = \mathbb{P}[\text{pick one dime from 1 penny, 1 nickel, 3 dimes, 4 quarters}] = \frac{3}{9}$

(d) Calculate the probability that the first coin is a nickel, given that the second coin is a dime. (4 pts)

Solution:

$\boxed{\text{Answer: } {}^{2}/_{9}}$

Total number of ordered selections without replacement of 2 coins out of 10 where second coin is a dime is: 9×3 (3 choices for dimes in second selection and 9 choices for first after leaving one dime out for second selection).

Total number of ordered selections without replacement of 2 coins out of 10 where first coin is a nickel and the second coin is a dime is: 2×3 (2 choices for nickel in first selection and 3 choices for dime in second selection).

Therefore,

$\mathbb{P}[\text{1st is nickel}|\text{2nd is dime}] = \frac{2\times3}{3\times9} = \frac{2}{9}$

Problem 3 *16 points*

You and your friend play ten games of chess. Each game results in a win, loss, or draw. For any given game, the probability that you win is 0.3, and the probability that you lose is 0.2. The outcomes of the ten games are all independent of each other.

(a) Let W be the total number of games that you win. Identify the family of PMFs to which the PMF of W belongs and its parameters. Be as specific as possible (e.g., if you think W is Poisson(λ), you should state the value of λ). (4 pts)

Solution:

$W \sim \text{Binomial}(10, 0.3)$

(b) Let M be the total number of decisive games (i.e., wins or losses, but not draws). Calculate $\mathsf{Var}[M]$. (4 pts)

Solution:

$\boxed{\text{Answer: } \mathsf{Var}[M] = 2.5}$

$\mathbb{P}[\text{decisive game}] = \mathbb{P}[\text{win}] + \mathbb{P}[\text{lose}] = 0.3 + 0.2 = 0.5\,.$

$M \sim \text{Binomial}(10, 0.5) \Rightarrow \mathsf{Var}[M] = 10 \times (0.5)(1 - 0.5) = 2.5$

(c) A win counts as 1 point; a draw, 0.5 points; a loss, 0 points. Let S be your final score. Calculate $\mathbb{E}[S]$. (4 pts)

Solution:

$\boxed{\text{Answer: } 5.5}$

Firstly, $\mathbb{P}[\text{draw}] = 1 - \mathbb{P}[\text{win}] - \mathbb{P}[\text{lose}] = 1 - 0.3 - 0.2 = 0.5\,.$

Method 1: Let $S_i =$ score in game number i. Since all games are independent and have identical probabilities of win, lose, or draw, the expected score in each game is the same and is given by $\mathbb{E}[S_i] = 1 \times \mathbb{P}[\text{win}] + 0.5 \times \mathbb{P}[\text{draw}] + 0 \times \mathbb{P}[\text{lose}] = 1 \times 0.3 + 0.5 \times 0.5 = 0.3 + 0.25 = 0.55$. Since there are 10 games in total, $\mathbb{E}[S] = 10 \times \mathbb{E}[S_1] = 10 \times 0.55 = 5.5\,.$

Method 2: Let D be the total number of draws. Then $D \sim \text{Binomial}(10, 0.5)$. The total score is $S = 1\times W + 0.5\times D = W + 0.5D \Rightarrow \mathbb{E}[S] = \mathbb{E}[W] + 0.5\mathbb{E}[D] = 10\times0.3 + 0.5\times10\times0.5 = 3 + 2.5 = 5.5\,.$

(d) You are distracted and end up losing all ten games. Your friend agrees to keep playing, but only until you win one game. Let N be the total number of games played till you win (including the original ten losses and the final win). Calculate $\mathbb{E}[N]$. (4 pts)

Solution:

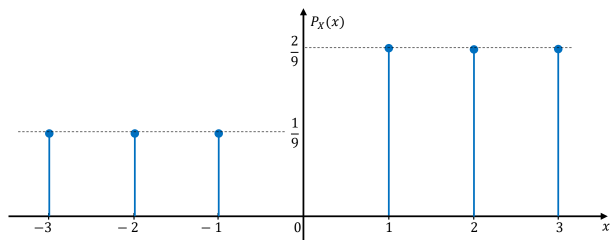
$\boxed{\text{Answer: } \mathbb{E}[N] = 13\tfrac{1}{3}}$

Let A = additional games after losing 10 games till you win. Then $A \sim$ Geometric(0.3) and $N = 10 + A$. Thus, $\mathbb{E}[N] = 10 + \mathbb{E}[A] = 10 + \frac{1}{0.3} = 13.3333$

Problem 4 *16 points*

Let X have the probability mass function with case-by-case formula below and plot shown on the right.

$$P_X(x) = \begin{cases} 1/9, & x = -3, -2, -1; \\ 2/9, & x = 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$



(a) Calculate $\mathbb{E}[X]$ and $\mathsf{Var}[X]$. (4 pts)

Solution:

$\boxed{\text{Answers: } \mathbb{E}[X] = \tfrac{2}{3}, \mathsf{Var}[X] = \tfrac{38}{9} \approx 4.22}$

$$\begin{aligned} \mathbb{E}[X] &= \frac{1}{9}(-3-2-1) + \frac{2}{9}(1+2+3) = (1+2+3)\frac{-1+2}{9} = \frac{6}{9} = \frac{2}{3} \\ \mathbb{E}[X^2] &= \frac{1}{9}((-3)^2 + (-2)^2 + (-1)^2) + \frac{2}{9}(1^2 + 2^2 + 3^2) = (1+4+9)\frac{1+2}{9} = \frac{14}{3} = 4\tfrac{1}{3} \approx 4.33 \\ \mathsf{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{14}{3} - \left(\frac{2}{3}\right)^2 = \frac{14}{3} - \frac{4}{9} = \frac{42-4}{9} = \frac{38}{9} = 4\tfrac{2}{9} \approx 4.22 \end{aligned}$$

(b) Let $Y = \begin{cases} 1 & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Compute $\mathsf{Var}[Y]$. (4 pts)

Solution:

$\boxed{\text{Answer: } \mathsf{Var}[Y] = \tfrac{2}{9}}$

Y is a Bernoulli(p) RV with $p = \mathbb{P}[Y = 1] = \mathbb{P}[X \in \{1, 2, 3\}] = 3 \times \frac{2}{9} = \frac{2}{3}$. Therefore, $\mathsf{Var}[Y] = p(1-p) = (2/3)(1-2/3) = \frac{2}{9}$.

(c) Let B be the event that $|X| < 2$. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula. (4 pts)

Solution:

$B = \{|X| < 2\} = \{X \in \{-1, 1\}\} \Rightarrow \mathbb{P}[B] = P_X(-1) + P_X(1) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$. We restrict and rescale to get the desired conditional PMF:

$$P_{X|B}(x) = \begin{cases} \frac{1/9}{3/9} = \frac{1}{3}, & x = -1; \\ \frac{2/9}{3/9} = \frac{2}{3}, & x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Compute $\mathsf{Var}[X \mid B]$, the variance of the random variable $X|B$ distributed according to the PMF $P_{X|B}(x)$ in the previous part. (4 pts)

Solution:

$\boxed{\text{Answer: } \mathsf{Var}[X|B] = \frac{8}{9}}$

Note: $\mathbb{E}[X^2|B] = 1$, since $R_{X|B} = \{-1, 1\}$. Also, $\mathbb{E}[X|B] = 1 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{1}{3}$. Then,

$$\mathsf{Var}[X|B] = \mathbb{E}[X^2|B] - (\mathbb{E}[X|B])^2 = 1 - \frac{1}{9} = \frac{8}{9}.$$

Problem 5 *16 points*

We want to develop a probability model based on the results of two EK381 entrance surveys, summarized in the table on the right. The sample space Ω is all possible Engineering students at BU who register for EK381 in either Fall or, if not, in Spring of one academic year (not limited to these two surveys). We assume that students do not register in both semesters.

Let S be a random variable which is 1 if a random student drawn from Ω takes EK381 in the Fall, and 0 if that student takes EK381 in the Spring and let events $B, E, C,$ and M represent whether the student is a BME, EE, CE, or ME major, respectively.

Registrations in EK381

Fall 2024/Spring 2025

	Semester		
Major	Fall	Spring	Total
BME	96	33	129
CE	46	18	64
EE	19	36	55
ME	54	94	148
Totals	215	181	396

(a) To which family does the PMF of S belong? Estimate its parameters using data in the table. (4 pts)

Solution:

$\boxed{\text{Answers: } S \sim \text{Bernoulli}\left(\frac{215}{396}\right).}$

(b) To which family does the conditional PMF of S given B, i.e., $P_{S|B}(x)$, belong? Estimate its parameters using data in the table. (4 pts)

Solution:

$\boxed{\text{Answer: } S|B \sim \text{Bernoulli}\left(\frac{96}{129}\right)}$

(c) If an instructor walks up to a random person in this exam room, what is the probability that the student is an ME major? Express this as a conditional probability and estimate its value using the table. (4 pts)

Solution:

$\boxed{\text{Answer: } \mathbb{P}[M|S=0] = \frac{94}{181}}$

(d) Suppose we model the number of Fall ME students as $\text{Binomial}\left(150, \frac{50}{150}\right)$ and suppose $\mathbb{P}[100 \text{ Fall ME students}] = 2^m \times \mathbb{P}[50 \text{ Fall ME students}]$. Calculate m and comment on the likelihood of a flipped enrollment (100 ME students in the Fall versus 50). Note: $\binom{n}{k} = \binom{n}{n-k}$. (4 pts)

Solution:

$\boxed{\text{Answer: } m = -50\text{. The chance of a flipped enrollment is extremely unlikely.}}$

$$2^m = \frac{\binom{150}{100}\left(\frac{1}{3}\right)^{100}\left(\frac{2}{3}\right)^{50}}{\binom{150}{50}\left(\frac{1}{3}\right)^{50}\left(\frac{2}{3}\right)^{100}} = \frac{\frac{2^{50}}{3^{150}}}{\frac{2^{100}}{3^{150}}} = \frac{2^{50}}{2^{100}} = 2^{-50}.$$

Since $2^{-50} = (2^{10})^{-5} = (1024)^{-5} < (10^3)^{-5} = 10^{-15}$, the chance of having a flipped enrollment (100 ME in the Fall versus 50) is extremely unlikely.

Problem 6 *12 points*

To improve sales, a cereal company places one coupon inside each box of cereal and offers a small prize to anyone who collects and submits 5 coupons all of different colors (no two have the same color). The coupon colors are assigned independently to each box and they are equally likely to be either red, green, blue, white, or orange (5 different colors). You have an unlimited number of cereal boxes to choose from.

(a) After buying the first box of cereal, what is the expected number of *additional* boxes you would need to buy to find a coupon of a different color than the first? (4 pts)

Solution:

$\boxed{\text{Answer: } 5/4 = 1.25}$ Let N_2 denote the number of boxes you would need to buy after the first to get a different colored coupon than the first. Then N_2 is a Geometric RV with parameter $p_2 = 4/5$ since to get a different color than the first, you would need a coupon having one of the remaining $5-1=4$ colors.

(b) After you collect coupons of two different colors, what is the expected number of *additional* boxes you would need to buy to find a coupon of a third color? (4 pts)

Solution:

$\boxed{\text{Answer: } 5/3 \approx 1.67}$ Let N_3 denote the number of boxes you would need to buy after collecting two colors in order to get a third color. Then N_3 is a Geometric RV with parameter $p_3 = 3/5$ since to get a third color, you would need a coupon having one of the remaining $5-2=3$ colors.

(c) What is the expected number of *total* boxes you would need to buy to win a prize? (4 pts)

Solution:

$\boxed{\text{Answer: } 5\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)}$ Let N_i denote the number of boxes you would need to buy after collecting $(i-1)$ different colors in order to get a new color. Then N_i is a Geometric RV with parameter $p_i = (6-i)/5$ since to get a new color, you would need a coupon having one of the remaining $5-(i-1) = 6-i$ colors. The total number of boxes you would need to buy to collect all

5 colors is $N_1 + N_2 + N_3 + N_4 + N_5$ and its expected value is equal to $\sum_{i=1}^{5} \mathbb{E}[N_i] = \sum_{i=1}^{5} \frac{5}{6-i} = 5\left(\sum_{j=1}^{5} \frac{1}{j}\right)$.

Problem 7 *6 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If X is Bernoulli(p) and $Y = 2X - 1$, then $\mathbb{E}[Y^2] = 1$. (2 pts)

Solution:

True. The range of X is $R_X = \{0, 1\} \Rightarrow$ range of Y is $R_Y = \{-1, 1\} \Rightarrow Y^2 = 1$.

(b) If X has $\mathbb{E}[X] = 5$ and standard deviation 5, then X could be a Poisson RV. (2 pts)

Solution:

False. $\mathsf{Var}[X] = 25 \neq \mathbb{E}[X] = 5$: For a Poisson RV the mean should equal the variance.

(c) If X is Uniform(a, b) and $P_X(0) = 0.5$, then $b - a = 2$. (2 pts)

Solution:

False. For all x in the range of a Uniform(a, b) RV, $\mathbb{P}_X(x) = \frac{1}{b-a+1} \Rightarrow b-a+1 = \frac{1}{0.5} \Rightarrow (b-a) = 1$.

Problem 8 *6 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that the sample space is denoted by Ω, that A, B, and C are events with $\mathbb{P}[A], \mathbb{P}[B], \mathbb{P}[C] \in (0, 1]$.

(a) If $0 < \mathbb{P}[B] < 1$, then $\mathbb{P}[A|B] + \mathbb{P}[A|B^c] = 1$ for any A. (2 pts)

Solution:

False. If $A = \Omega$ then $\mathbb{P}[A|B] = \mathbb{P}[A|B^c] = 1$.

(b) If $\mathbb{P}[A] = 0.8$ and $\mathbb{P}[B|A] = 0.5$, then A and $(A \cap B)$ must be independent. (2 pts)

Solution:

False. $A \cap B$ is a subset of A, and so $\mathbb{P}[A|A \cap B] = 1$ which is not equal to $\mathbb{P}[A]$.

(c) If $\mathbb{P}[A] = 0.8$ and $\mathbb{P}[B|A] = 0.5$, then $\mathbb{P}[A] > \mathbb{P}[B]$. (2 pts)

Solution:

True. $\mathbb{P}[A] = 0.8 \Rightarrow \mathbb{P}[A^c] = 0.2$. Therefore, $\mathbb{P}[B] = \mathbb{P}[A]\mathbb{P}[B|A] + \mathbb{P}[A^c]\mathbb{P}[B|A^c] = (0.8)(0.5) + (0.2)\mathbb{P}[B|A^c] = 0.4 + 0.2\mathbb{P}[B|A^c] \leq 0.4 + 0.2 = 0.6 < 0.8 = \mathbb{P}[A]$.