Exam 1 Solutions

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$. The sample space is denoted by Ω.

(a) Suppose $\mathbb{P}[A] = 0.5, \mathbb{P}[B] = 0.5$ and and $\mathbb{P}[A \cup B] = 1$. Then, A and B cannot be independent.

Solution:

True. Note that $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = 1 - \mathbb{P}[A \cap B] = 1$, so $\mathbb{P}[A \cap B] = 0 \neq \mathbb{P}[A] \cdot \mathbb{P}[B]$.

(b) $\mathbb{P}[A|B \cap C] \leq \min\{\mathbb{P}[A|B], \mathbb{P}[A|C]\}$ (equivalently, $\mathbb{P}[A|B \cap C] \leq \mathbb{P}[A|B]$ and $\mathbb{P}[A|B \cap C] \leq \mathbb{P}[A|C]$).

Solution:

False. As a counterexample, let $A = 2$, $B = 1,2$, and $C = 2,3$. Also, let $\mathbb{P}[\{1\}] = \mathbb{P}[\{2\}] = \mathbb{P}[\{3\}] = 1/3$. Since $A = B \cap C$, we have that $\mathbb{P}[A|B \cap C] = 1$. However, $P[A|B] = 1/2, \mathbb{P}[A|C] = 1/2$.

(c) $\mathbb{P}(A \cap B) + \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Solution:

True. This follows from $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) If A, B, C are independent, then $\mathbb{P}[A \cup B \cup C] = 1 - \mathbb{P}[A]\,\mathbb{P}[B]\,\mathbb{P}[C]$.

Solution:

False.

$$\mathbb{P}[A \cup B \cup C] = 1 - \mathbb{P}[(A \cup B \cup C)^c] = 1 - \mathbb{P}[A^c \cap B^c \cap C^c] = 1 - \mathbb{P}[A^c]\mathbb{P}[B^c]\mathbb{P}[C^c]$$

(e) If A, B, and C are events such that $\mathbb{P}[A \mid B] > \mathbb{P}[A]$ and $\mathbb{P}[B \mid C] > \mathbb{P}[B]$, then $\mathbb{P}[A \mid C] > \mathbb{P}[A]$.

Solution:

The statement is tantalizing but false. For example, let the sample space be the unit interval with the probability assigned to an event equal to its length, and let $A = [\frac{1}{16}, \frac{1}{4}]$, $B = [\frac{1}{8}, \frac{1}{2}]$, and $C = [\frac{1}{4}, \frac{9}{16}]$. Then

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{1/8}{3/8} = \frac{1}{3} > \frac{3}{16} = \mathbb{P}[A]$$

and

$$\mathbb{P}[B \mid C] = \frac{\mathbb{P}[B \cap C]}{\mathbb{P}[C]} = \frac{1/4}{5/16} = \frac{4}{5} > \frac{3}{8} = \mathbb{P}[B].$$

However $\mathbb{P}[A \mid C] = 0$, which is less than $\mathbb{P}[A] = \frac{3}{16}$.

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If for some constant c we have $\mathbb{P}[\{X > c\}] = 1$, then $\mathbb{E}[X] > c$.

Solution:

True. All values taken by X are larger than c, and the expected value is a weighted average of the values taken by X.

(b) If $F_X(0) > 1/2$, then $\mathbb{E}[X] \leq 0$.

Solution:

False. $P_X(x) = \begin{cases} 3/4 & x = 0, \\ 1/4 & x = 1 \end{cases} \implies F_X(0) = 3/4$ but $\mathbb{E}[X] = 1/4 > 0$

(c) If $F_X(3) = F_X(2)$, then $P_X(3) = 0$.

Solution:

True, since otherwise $F_X(3)$ has a jump of size $P_X(3)$ at $x = 3$.

(d) If X is a Geometric($\frac{1}{4}$) random variable, then $\mathbb{E}[\ln(X)] > 0$.

Solution:

True, since $R_X = \{1, 2, \ldots\}$, and so $\ln(x) > 0$ for all values in R_X except for 1, where $\ln(X) = 0$.

(e) If $Y = X + 2$, then $F_Y(a) = F_X(a - 2)$.

Solution:

True. $F_Y(a) = \mathbb{P}[\{Y \leq a\}] = \mathbb{P}[\{X + 2 \leq a\}] = \mathbb{P}[\{X \leq a - 2\}] = F_X(a - 2)$.

Problem 3 Complete the following quick calculations. *16 points*

(a) Let X be Discrete Uniform $(1, b)$, and let $\mathbb{E}[X] = 5$. Compute b and $\mathsf{Var}[X]$.

Solution:

Since $\mathbb{E}[X] = \frac{1+b}{2} = 5$, then $b = 10 - 1 = 9$. From the formula for the variance of a discrete uniform random variable,

$$\mathsf{Var}[X] = \frac{(b-a)(b-a+2)}{12} = \frac{8 \times 10}{12} = \frac{20}{3}.$$

(b) Let X be Poisson (3). Calculate $\mathbb{E}[X + 1]$ and $\mathsf{Var}[3 - 2X]$.

Solution:

For Poisson (λ) RVs, $\mathbb{E}[X] = \lambda, \mathsf{Var}[X] = \lambda$. Hence,

$$\mathbb{E}[X+1] = 1 + \mathbb{E}[X] = 4;\ \mathbb{E}[X^2] = \mathsf{Var}[X] + (\mathbb{E}[X])^2 = 3 + 3^2 = 12$$

So,

$$\mathsf{Var}[3-2X] = \mathsf{Var}[-2X] = 4\mathsf{Var}[X] = 12.$$

(c) Let X be a Bernoulli $(\frac{1}{3})$ random variable. Compute $\mathbb{E}[3-2X]$ and $\mathbb{E}[2^X]$.

Solution:

For Bernoulli (p) RVs, $R_X = \{0,1\}$, and $P_X(x) = p$ if $x = 1$, and 0 elsewhere. Thus

$$\mathbb{E}[3-2X] = 3 - 2\mathbb{E}[X] = 3 - 2\frac{1}{3} = \frac{7}{3}.$$

$$\mathbb{E}[2^X] = \sum_{x \in R_X} 2^x P_X(x) = \frac{2}{3}2^0 + \frac{1}{3}2 = \frac{2}{3} + \frac{1}{3}2 = \frac{4}{3}.$$

(d) Let A and B be events with $\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[A \cap B] = \frac{3}{8}$ and $\mathbb{P}[A^{\mathsf{c}} \cap B] = \frac{1}{8}$. Calculate $\mathbb{P}[B]$ and $\mathbb{P}[A|B^c]$.

Solution:

By the total probability theorem,

$$\mathbb{P}[B] = \mathbb{P}[B \cap A] + \mathbb{P}[B \cap A^c] = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\mathbb{P}[A|B^c = \frac{\mathbb{P}[A \cap B^c]}{\mathbb{P}[B^c]}$$

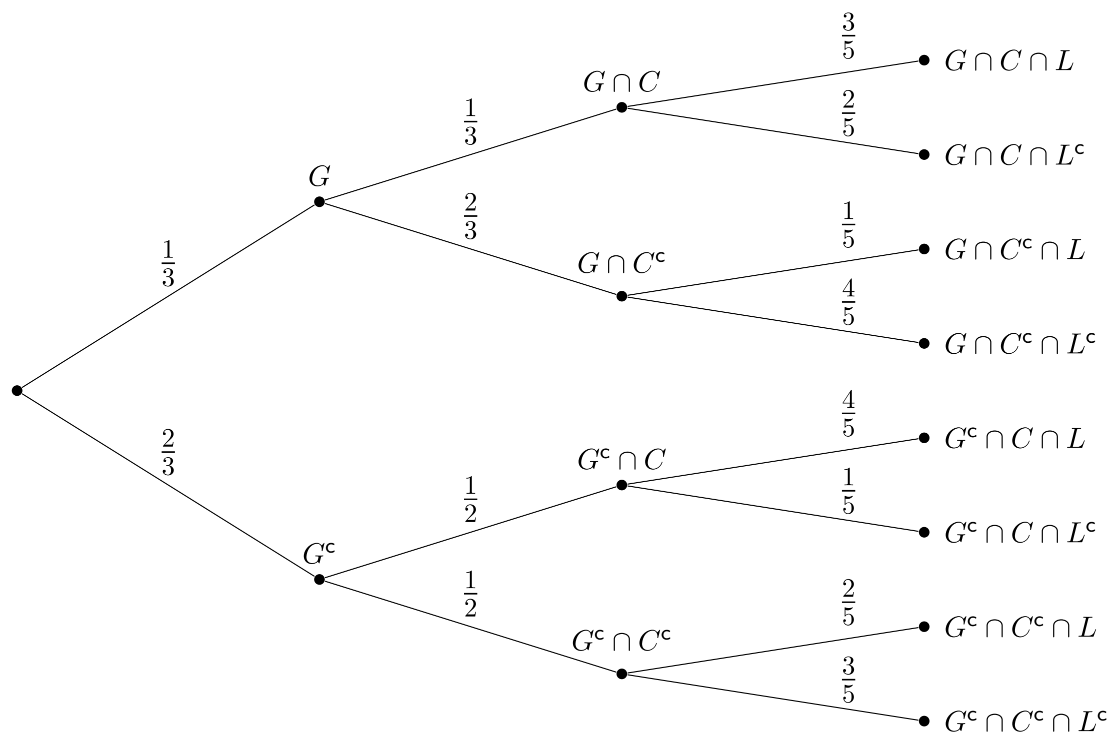
$$\mathbb{P}[A \cap B^c] + \mathbb{P}[A \cap B] = \mathbb{P}[A] \implies \mathbb{P}[A \cap B^c] = \frac{1}{8}$$

Thus, $\mathbb{P}[B] = 1/2$, so $\mathbb{P}[B^c] = 1/2$. Furthermore,

$$\mathbb{P}[A|B^c] = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

Problem 4 *16 points*

You always take the same bus to school and have built a probability model to predict when you will be late. Specifically, you have made the following conditional probability tree where G is the event that the weather is good, C is the event that the bus is crowded, and L the event that you are late to class.



(a) What is the probability that the weather is good and you are late to class?

Solution:

We can see from the tree that there are two mutually exclusive events $G \cap C \cap L$ and $G \cap C^{\mathsf{c}} \cap L$ that contain $G \cap L$, so we can just add up their probabilities, which we obtain from the tree using the multiplication rule,

$$\mathbb{P}[G \cap L] = \mathbb{P}[G \cap C \cap L] + \mathbb{P}[G \cap C^{\mathsf{c}} \cap L] = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{5} = \frac{1}{9}.$$

(b) Given that the weather is good, what is the probability of being late to class?

Solution:

Using part (a), we can use the definition of conditional probability to get

$$\mathbb{P}[L|G] = \frac{\mathbb{P}[L \cap G]}{\mathbb{P}[G]} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}.$$

We can also obtain this directly from the conditional probability tree, starting at G and adding up the probabilities to each of the leaves that contain L:

$$\mathbb{P}[L|G] = \mathbb{P}[C|G]\,\mathbb{P}[L|G \cap C] + \mathbb{P}[C^{\mathsf{c}}|G]\,\mathbb{P}[L|G \cap C^{\mathsf{c}}] = \frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{5} = \frac{1}{3}.$$

You can also think of this as a conditional version of the Total Probability Theorem.

(c) What is the probability of being late to class?

Solution:

We can express this using the Total Probability Theorem as

$$\mathbb{P}[L] = \mathbb{P}[G]\,\mathbb{P}[L|G] + \mathbb{P}[G^{\mathsf{c}}]\,\mathbb{P}[L|G^{\mathsf{c}}]$$

so that we can take advantage of the fact that we have $\mathbb{P}[L|G]$ from part (b). Using the same approach as in (a), we obtain

$$\mathbb{P}[L|G^{\mathsf{c}}] = \mathbb{P}[C|G^{\mathsf{c}}]\,\mathbb{P}[L|G^{\mathsf{c}} \cap C] + \mathbb{P}[C^{\mathsf{c}}|G^{\mathsf{c}}]\,\mathbb{P}[L|G^{\mathsf{c}} \cap C^{\mathsf{c}}] = \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}.$$

Now, plugging everything in, we get $\mathbb{P}[L] = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{3}{5} = \frac{23}{45}$.

(d) Given that you are late to class, what is the probability that the bus was crowded?

Solution:

We can get this by directly applying the definition of conditional probability, writing the intersection $C \cap L$ in terms of the leaves of the tree, $C \cap L = (G \cap C \cap L) \cup (G^{\mathsf{c}} \cap C \cap L)$, and multiplying through to evaluate these probabilities (i.e., the multiplication rule):

$$\mathbb{P}[C|L] = \frac{\mathbb{P}[C \cap L]}{\mathbb{P}[L]} = \frac{\mathbb{P}[G \cap C \cap L] + \mathbb{P}[G^{\mathsf{c}} \cap C \cap L]}{\mathbb{P}[L]} = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{4}{5}}{\frac{23}{45}} = \frac{\frac{1}{3}}{\frac{23}{45}} = \frac{15}{23}.$$

Problem 5 *12 points*

Consider the following simple lottery game. There are ten balls total of which five are red and numbered $1, 2, 3, 4, 5$. The remaining five are blue and also numbered $1, 2, 3, 4, 5$. Three balls are simultaneously pulled out from a hat, and depending on what they are, you might win a prize. Useful facts: $\binom{10}{3} = 120$; $\binom{5}{3} = 10$.

(a) You win a teddy bear if all three balls are the same color. What is the probability of winning a teddy bear?

Solution:

There are 2 colors and, for each color, there are $\binom{5}{3} = \frac{5!}{3!\;2!} = 10$ ways to select 3 out of 5 balls. Thus, there are $2 \cdot 10 = 20$ ways to win a teddy bear.

$$\mathbb{P}[\text{Teddy}] = \frac{\#\text{ ways to win Teddy}}{\#\text{ total ways to draw}} = \frac{20}{120} = \frac{1}{6}.$$

(b) You decide to play the same lottery game for 4 days in a row (and each day's outcome is independent of the others). What is the probability that you win *at least one* teddy bear?

Solution:

Let X be the total number of teddy bears you win after 4 days and note that $X \sim \text{Binomial}(4, 1/6)$.

Therefore, we can write

$$\mathbb{P}[\{\text{at least 1 teddy}\}] = 1 - \mathbb{P}[X=0] = 1 - (\frac{5}{6})^4$$
$$= 1 - \frac{625}{1296} = \frac{671}{1296}.$$

(c) In the same lottery, you can also win a Saint Seiya figurine if the three balls have consecutive numbers (such as red 2, blue 3, and blue 4) regardless of color (or order). What is the probability of winning a Saint Seiya?

Solution:

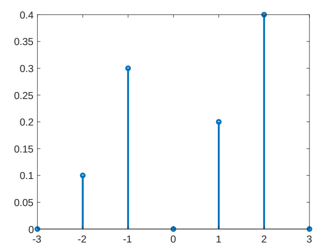
You can either win with balls numbered $1,2,3$ or $2,3,4$ or $3,4,5$. For each of these sequences, each ball can be either red or blue. That means that there are $2\times2\times2=8$ ways to draw each sequence, meaning there are $3\times8=24$ ways to win. We are drawing 3 out of 10 balls without replacement and with order independence. Thus, there are $\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10\cdot9\cdot8}{3\cdot2\cdot1} = 120$ ways to draw. Since all outcomes are equally likely,

$$\mathbb{P}[\text{Saint Seiya}] = \frac{\#\text{ ways to win Saint Seiya}}{\#\text{ total ways to draw}} = \frac{24}{120} = \frac{1}{5}.$$

Problem 6 *16 points*

Let X have the probability mass function below, with plot shown on the right.

$$P_X(x) = \begin{cases} 1/10, & x=-2; \\ 3/10, & x=-1; \\ 2/10, & x=1; \\ 4/10, & x=2; \\ 0, & \text{otherwise.} \end{cases}$$



(a) Calculate $\mathbb{E}[X]$ and $\mathsf{Var}[X]$.

Solution:

$$\mathbb{E}[X] = (-2)\frac{1}{10} + (-1)\frac{3}{10} + (1)\frac{2}{10} + (2)\frac{4}{10} = \frac{1}{2}$$
$$\mathbb{E}[X^2] = (-2)^2\frac{1}{10} + (-1)^2\frac{3}{10} + (1)^2\frac{2}{10} + (2)^2\frac{4}{10} = \frac{4+3+2+16}{10} = \frac{25}{10}$$
$$\mathsf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{25}{10} - (\frac{1}{2})^2 = \frac{45}{20} = \frac{9}{4}$$

(b) Let B be the event that $|X| < 2$. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula.

Solution:

$\mathbb{P}[B] = P_X(1) + P_X(-1) = \frac{5}{10}$. We restrict and rescale to get the desired conditional PMF:

$$P_{X|B}(x) = \begin{cases} \frac{3/10}{5/10} = \frac{3}{5}, & x = -1; \\ \frac{2/10}{5/10} = \frac{2}{5}, & x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Compute $\mathsf{Var}[X \mid B]$, the variance of the random variable $X|B$ distributed according to the PMF $P_{X|B}(x)$ in the previous part.

Solution:

Note $\mathbb{E}[X^2|B] = 1$, since $R_{X|B} = \{-1, 1\}$. Also, $\mathbb{E}[X|B] = \frac{2}{5} - \frac{3}{5} = -\frac{1}{5}$. Then,

$$\mathsf{Var}[X|B] = \mathbb{E}[X^2|B] - (\mathbb{E}[X|B])^2 = 1 - \frac{1}{25} = \frac{24}{25}.$$

(d) Compute $\mathbb{E}\left[\frac{1}{X^3} \mid B\right]$.

Solution:

$$\begin{aligned} \mathbb{E}\left[\frac{1}{X^3} \mid B\right] &= \sum_{x \in \{-1,1\}} \frac{1}{x^3} P_{X|B}(x) \\ &= (-1)^3 P_{X|B}(-1) + (1)^3 P_{X|B}(1) \\ &= -\frac{3}{5} + +1 \cdot \frac{2}{5} = -\frac{1}{5} \end{aligned}$$

Problem 7 *16 points*

A service center receives a random number X of service requests per day, which we model as a Binomial(5,1/4) random variable. The number of service requests is independent from day to day. Useful formulas: $\binom{5}{2} = \binom{5}{3} = 10$; $\binom{5}{1} = \binom{5}{4} = 1$. Your answers should not contain combinations.

(a) What is the probability of getting either two or three service requests in a day?

Solution:

$$P_X(2) + P_X(3) = \binom{5}{2}(\frac{1}{4})^2(\frac{3}{4})^3 + \binom{5}{3}(\frac{1}{4})^3(\frac{3}{4})^2 = 10(\frac{1}{4})^2(\frac{3}{4})^3 + 10(\frac{1}{4})^3(\frac{3}{4})^2 = \frac{360}{1024} = \frac{45}{128}$$

(b) Given that, on a given day, you know the center received no more than two requests, what is the probability it has only one request?

Solution:

The conditioning event B, no more than two requests, has $\mathbb{P}[B] = P_X(0) + P_X(1) + P_X(2)$.

$$P_X(1|B) = \frac{P_X(1)}{P_X(0)+P_X(1)+P_X(2)} = \frac{3^4\times 5}{3^5+3^4\times 5+3^3\times 10} = \frac{3\times 5}{3^2+3\times 5+10} = \frac{15}{34}.$$

(c) What is the probability that, on a given day, the service center receives the maximum number of five requests?

Solution:

$P_X(5) = \frac{1}{4^5} = \frac{1}{2^{10}} = \frac{1}{1024}$ is the probability that, on any given day, you have 5 service requests.

(d) What is the expected number of days until the service center receives five requests in a day?

Solution:

The random variable Y consisting of the first day where you get five requests in a day is a Geometric($\frac{1}{1024}$) random variable. Then,

$$\mathbb{E}[Y] = 4^5 = 1024$$

Problem 8 *3 points*

A black dog and a brown dog are going to have a litter of puppies. The number of puppies in the litter is a Discrete Uniform(2,6) random variable. For any size litter, the probability that a puppy will be black is $\frac{3}{4}$ and brown $\frac{1}{4}$, and the colors of the puppies are independent across the litter.

(a) (1 pt) What is the expected number of black puppies?

Solution:

Remarkably, it is simply the average number of puppies (4) times $\frac{3}{4}$, which is 3.

(b) (3 pts) What is the probability that we get at least 4 black puppies?

Solution:

Let A be the event of four or more black puppies in a litter. Let $B_k, k = 2, 3, \ldots, 6$ be the events that the litter is of size k.

Note that $B_2, B_3, \ldots, B_6$ is a partition. Then, by the total probability theorem,

$$\mathbb{P}[A] = \mathbb{P}[A \mid B_2]\mathbb{P}[B_2] + \mathbb{P}[A \mid B_3]\mathbb{P}[B_3] + \mathbb{P}[A \mid B_4]\mathbb{P}[B_4] + \mathbb{P}[A \mid B_5]\mathbb{P}[B_5] + \mathbb{P}[A \mid B_6]\mathbb{P}[B_6]$$

Note that you must have a litter of at least 4 puppies to get 4 black puppies. Hence, $\mathbb{P}[A \mid B_2] = \mathbb{P}[A \mid B_3] = 0$, so

$$\mathbb{P}[A] = \mathbb{P}[A \mid B_4]\mathbb{P}[B_4] + \mathbb{P}[A \mid B_5]\mathbb{P}[B_5] + \mathbb{P}[A \mid B_6]\mathbb{P}[B_6]$$

Let X be the number of black puppies in a litter. Then, $P_{X|B_4}$ is Binomial$(4, 3/4)$, and thus $\mathbb{P}[A|B_4] = P_{X|B_4}(4) = (\frac{3}{4})^4$. $P_{X|B_5}$ is Binomial$(5, 3/4)$, so

$$\mathbb{P}[A|B_5] = P_{X|B_5}(4) + P_{X|B_5}(5) = 5(\frac{3}{4})^4(\frac{1}{4}) + (\frac{3}{4})^5$$

Similarly, $P_{X|B_6}$ is Binomial$(6, 3/4)$, so

$$\mathbb{P}[A|B_6] = P_{X|B_6}(4) + P_{X|B_6}(5) + P_{X|B_6}(6) = 15(\frac{3}{4})^4(\frac{1}{4}))^2 + 6(\frac{3}{4})^5(\frac{1}{4})) + (\frac{3}{4})^6$$

Hence,

$$\mathbb{P}[A] = (\frac{1}{5})(\frac{3^4}{4^6})(16 + 20 + 12 + 15 + 18 + 9) = \frac{3^6}{2^{11}} = \frac{729}{2048}$$