Exam 1

Last Name	First Name	Student ID #
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ____________________

Partial Credit: The most important issue is knowing how to approach a particular problem. Therefore, there will be partial credit for good solution outlines even if not all the mathematical manipulations are completed correctly. Be sure to attempt every problem!

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed** - including no phones and no calculators.
- You can use the provided formula sheet handout - no extra materials are allowed.
- No form of collaboration is allowed.
- All work to be graded must be included in this document. Submit no extra sheets.
- There are 8 problems in total, with different point values. The last problem is *worth little, but is conceptually harder*. Make sure you attempt all other problems before this one.

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		10	Problem 5		15
Problem 2		10	Problem 6		15
Problem 3		16	Problem 7		15
Problem 4		15	Problem 8		4
			Total		100

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)(1 - \mathbb{P}(B^c))$

Solution:

True. $\mathbb{P}[A|B] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A\cap B]}{1-\mathbb{P}[B^c]}$.

(b) If A and B are independent, $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]\mathbb{P}[A^c]$.

Solution:

True.
$$\begin{aligned} P[A \cup B] = P[A] + P[B] - P[A \cap B] &= P[A] + P[B] - P[A]P[B] \\ &= P[A] + P[B]\big(1 - P[A]\big) \\ &= P[A] + P[B]P[A^c] \end{aligned}$$

(c) If $A \subset B$, then $\mathbb{P}[A|C] \leq \mathbb{P}[B|C]$.

Solution:

True. $\mathbb{P}[A|C] = \dfrac{\mathbb{P}[A \cap C]}{\mathbb{P}[C]}$ and $\mathbb{P}[B|C] = \dfrac{\mathbb{P}[B \cap C]}{\mathbb{P}[C]}$. Now, we can write $B \cap C = (A \cap C) \cup (A^{\mathsf{c}} \cap B \cap C)$ where $A \cap C$ and $A^{\mathsf{c}} \cap B \cap C$ are mutually exclusive. Thus, by the additivity axiom, $\mathbb{P}[B \cap C] = \mathbb{P}[A \cap C] + \mathbb{P}[A^{\mathsf{c}} \cap B \cap C] \geq \mathbb{P}[A \cap C]$, which implies that $\mathbb{P}[B|C] \geq P[A|C]$.

(d) If $A \cup B = \Omega$, where Ω is the sample space, then $\mathbb{P}[A] + \mathbb{P}[B] = 1$.

Solution:

False. It would be true if A, B were disjoint, but not in general.

(e) $\mathbb{P}[A \mid B] \geq \mathbb{P}[A]$.

Solution:

False. If A, B are disjoint and $\mathbb{P}[A] > 0, \mathbb{P}[B] > 0$, we have $\mathbb{P}[A \mid B] = 0$.

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $F_X(1) = F_X(3)$, then $P_X(2) = 0$.

> **Solution:**
>
> True. This tells us that $P[X \leq 1] = P[X \leq 3]$, then $P_X(2) = P[X = 2] = 0$.

(b) $\mathsf{Var}(3 - X) = \mathsf{Var}(X)$.

> **Solution:**
>
> True. $\mathsf{Var}(aX + b) = a^2\mathsf{Var}(X)$.

(c) For any random variable X, $\mathbb{E}[X^3] = (\mathbb{E}[X])^3$.

> **Solution:**
>
> Clearly false, as X^3 is nonlinear.

(d) Let X be a Binomial$(10, 1/4)$ random variable and Y be a Binomial$(5, 1/4)$ random variable. Then, $X + Y$ is a Binomial$(15, 1/4)$ random variable.

> **Solution:**
>
> True as intended. The idea is that the random number of successes in 10 attempts, added to the random number of successes in 5 attempts with the same probability of success, will be the random number of successes in 15 total attempts. One is the number of successes in 10 independent trials with probability of success $1/4$, and the other is the number of successes in 5 independent trials with the same probability of success. Note that the PMF of the sum of random variables is not the sum of their PMFs.
>
> However, the question could have different interpretations. False was accepted provided a good explanation was given.

(e) If $x \in B$, and $\mathbb{P}[B] > 0$, then $P_{X|B}(x) \geq P_X(x)$.

> **Solution:**
>
> True. $P_{X|B}(x) = \frac{P_X(x)}{\mathbb{P}[B]}$, and $\mathbb{P}[B] \leq 1$.

Problem 3 Please complete the following quick calculations. *16 points*

(a) Suppose that out of the students in a class, 60% are geniuses (event G), 70% love chocolate (event C), and 40% fall into both categories (event $G \cap C$). Determine the probability that a randomly selected student is neither a genius nor a chocolate lover ($\mathbb{P}[(G \cup C)^c)]$, and the probability that a genius loves chocolate ($\mathbb{P}[C|G]$).

Solution:

We know $\mathbb{P}[G] = 0.6, \mathbb{P}[C] = 0.7, \mathbb{P}[G \cap C] = 0.4$. Hence,

$$\mathbb{P}[G \cup C] = 0.6 + 0.7 - 0.4 = 0.9$$

so $\mathbb{P}[(G \cup C)^c)] = 1 - \mathbb{P}[G \cup C] = 0.1$.

$$\mathbb{P}[C|G] = \frac{\mathbb{P}[C \cap G]}{\mathbb{P}[G]} = \frac{0.4}{0.6} = \frac{2}{3}$$

(b) Let X be Binomial$(7, \frac{1}{2})$. Calculate $\mathbb{P}[1 \le X \le 6]$ and $\mathbb{P}\big[X = 2 \,\big|\, 1 \le X \le 6\big]$. Useful facts: $2^7 = 128$; $\binom{7}{1} = \binom{7}{6} = 7$; $\binom{7}{2} = \binom{7}{5} = 21$; $\binom{7}{3} = \binom{7}{4} = 35$.

Solution:

$$\mathbb{P}[1 \le X \le 6] = 1 - \big(P_X(0) + P_X(7)\big) = 1 - \left(\binom{7}{0}\left(\frac{1}{2}\right)^2 + \binom{7}{7}\left(\frac{1}{2}\right)^2\right) = 1 - \frac{1}{2^6} = \frac{63}{64}$$

$$\mathbb{P}\big[X = 2 \,\big|\, 1 \le X \le 6\big] = \frac{\mathbb{P}\big[\{X = 2\} \cap \{1 \le X \le 6\}\big]}{\mathbb{P}[1 \le X \le 6]} = \frac{P_X(2)}{\mathbb{P}[1 \le X \le 6]} = \frac{\binom{7}{2}(\frac{1}{2})^7}{\frac{63}{64}} = \frac{\frac{21}{128}}{\frac{63}{64}} = \frac{1}{6}$$

(c) Let X be Discrete Uniform$(1, 37)$. Let B the event that X is strictly greater than 4 *and* strictly less than 8. Calculate $\mathbb{E}[X|B]$ and $\mathbb{E}[X^2|B]$.

Solution:

Note that the new conditional PMF $P_{X|B}(x)$ is Discrete Uniform$(5, 7)$ by restricting the PMF to the values in $B = \{5, 6, 7\}$ and rescaling, dividing by $\mathbb{P}[B] = \frac{3}{37}$. Therefore,

$$\mathbb{E}[X|B] = \frac{5+7}{2} = 6; \quad \mathsf{Var}[X|B] = \frac{(7-5+1)^2 - 1}{12} = \frac{8}{12} = \frac{2}{3}$$
$$\mathbb{E}[X^2|B] = \mathsf{Var}[X|B] + \big(\mathbb{E}[X|B]\big)^2 = \frac{2}{3} + 6^2 = \frac{110}{3}$$

(d) Let X be Poisson(2). Calculate $\mathbb{E}[3X - 1]$ and $\mathbb{E}[3 - X^2]$.

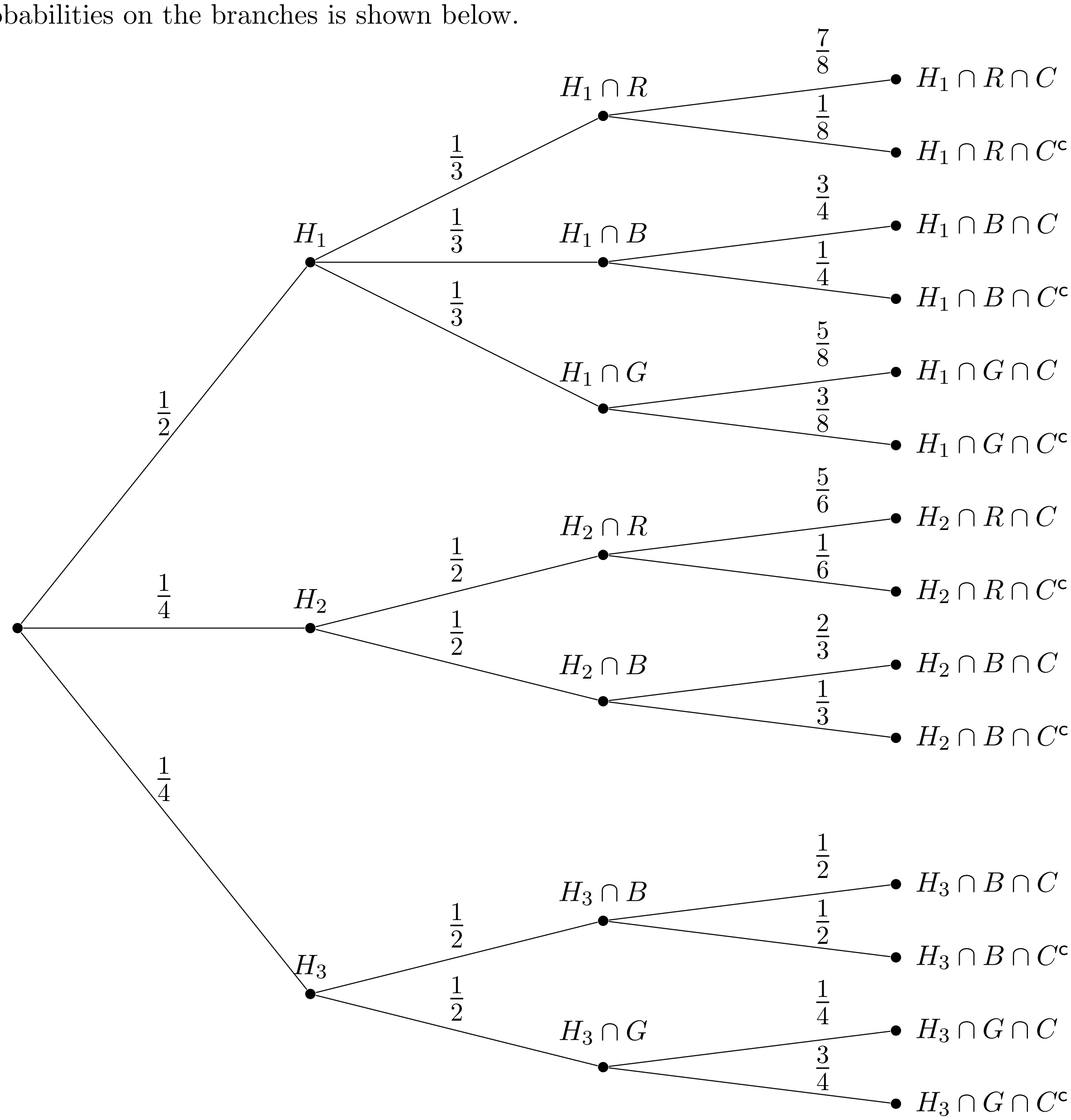
Solution:

$\mathbb{E}[3X - 1] = 3\mathbb{E}[X] - 1 = 3 \cdot 2 - 1 = 5$.

$$\mathbb{E}[3 - X^2] = 3 - \mathbb{E}[X^2] = 3 - ((\mathbb{E}[X])^2 + \mathsf{Var}(X)) = 3 - (4 + 2) = -3$$

Problem 4 *15 points*

You are playing a simple game where you first select one of three hats, and then reach into the hat and grab a box, which is either red, green, or blue, and then finally open the box to see if there is a chocolate inside. Specifically, let H_i be the event that you choose the i^{th} hat, R the event that you pick a red box, G the event that you pick a green box, B the event that you pick a blue box, and C the event that you find a chocolate. A tree diagram with conditional probabilities on the branches is shown below.



(a) What is the probability that you choose a blue box?

Solution:

$$\mathbb{P}[B] = \mathbb{P}[B \cap H_1] + \mathbb{P}[B \cap H_2] + \mathbb{P}[B \cap H_3] = 1/6 + 1/8 + 1/8 = 5/12.$$

(b) What is the probability that you choose a green box and do not find a chocolate?

Solution:

$\mathbb{P}[G \cap C^c] = \mathbb{P}[H_3 \cap G \cap C^c] + \mathbb{P}[H_1 \cap G \cap C^c] = 3/32 + 1/16 = 5/32.$

(c) Given that you choose a green box and do not find a chocolate, what is the probability that you selected the third hat?

Solution:

Using the answer from the previous part, $\mathbb{P}[H_3 | G \cap C^c] = \frac{\mathbb{P}[H_3 \cap G \cap C^c]}{\mathbb{P}[G \cap C^c]} = 3/5.$

(d) Given that you choose a blue box, what is the probability that you find a chocolate?

Solution:

$\mathbb{P}[C|B] = \frac{\mathbb{P}[B \cap C]}{\mathbb{P}[B]}$. We know the denominator from part (a). The numerator is

$\mathbb{P}[B \cap C] = \mathbb{P}[H_1 \cap B \cap C] + \mathbb{P}[H_2 \cap B \cap C] + \mathbb{P}[H_3 \cap B \cap C] = 1/8 + 1/12 + 1/16 = 13/48$

using the Law of Total Probability. Then, $\mathbb{P}[C|B] = 13/20$.

(e) Given that you select the first hat and find a chocolate, what is the probability you choose a blue box?

Solution:

By the definition of conditional probability, $\mathbb{P}[B|H_1 \cap C] = \frac{\mathbb{P}[H_1 \cap B \cap C]}{\mathbb{P}[H_1 \cap C]}$. The numerator is $1/8$.

The denominator is computed by the Law of Total Probability:

$\mathbb{P}[H_1 \cap C] = \mathbb{P}[H_1 \cap B \cap C] + \mathbb{P}[H_1 \cap R \cap C] + \mathbb{P}[H_1 \cap G \cap C] = 1/8 + 7/48 + 5/48 = 9/24.$

Thus, $\mathbb{P}[B|H_1 \cap C] = \frac{1/8}{9/24} = 1/3$.

Problem 5 *15 points*

The number of geese X you see on the Charles River Path on a stroll is modeled by a Poisson (λ) random variable. (Throughout the problem, you might encounter terms that include $e^{-\lambda}$ for some λ. You can leave these as is, since you should not use a calculator.)

(a) Say that you learn that $\mathbb{E}[5X] = 10$. Select the parameter λ based on this fact, and set it equal to this value for the remainder of the problem.

Solution:

For Poisson(λ), $\mathbb{E}[5X] = 5\lambda = 10$, so $\lambda = 2$.

(b) Is it more likely to observe 1 goose or 3 geese on your stroll?

Solution:

$\mathbb{P}_X(1) = 2e^{-2}$. $\mathbb{P}_X(3) = \frac{2^3}{6}e^{-2}$. 1 is more likely.

(c) Say that you learn that the number of geese you will observe on cloudy days is at most 3. Given that it is cloudy, what is the average number of geese you observe?

Solution:

$C = \{X \leq 3\}$. $\mathbb{P}[C] = (1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!})e^{-2} = \frac{19}{3}e^{-2}$. $P_{X|C}(x) = \begin{cases} \frac{3}{19} & x = 0 \\ \frac{6}{19} & x = 1 \\ \frac{6}{19} & x = 2 \\ \frac{4}{19} & x = 3 \end{cases}$

$$\mathbb{E}[X|C] = \frac{6}{19} + \frac{6}{19} \cdot 2 + \frac{4}{19} \cdot 3 = \frac{30}{13}.$$

(d) Given that it is cloudy, what is the probability you do not see any geese?

Solution:

$P_{X|C}(0) = \frac{3}{19}$.

(e) On a sunny day, someone tells you there is at least one goose on the path. Given this information, what is the probability that there are at least two geese on the path?

Solution:

$D = \{X \geq 1\}$. $\mathbb{P}[D] = 1 - \mathbb{P}[\{X = 0\}] = 1 - e^{-2}$.

$\mathbb{P}[\{X > 1\}|D] = 1 - \mathbb{P}[\{X = 1\}|D]$.

$\mathbb{P}[\{X = 1\}|D] = \frac{P_X(1)}{\mathbb{P}[D]} = \frac{2e^{-2}}{1-e^{-2}}$.

Hence, $\mathbb{P}[\{X > 1\}|D] = 1 - \frac{2e^{-2}}{1-e^{-2}} = \frac{1-3e^{-2}}{1-e^{-2}}$

Problem 6 *16 points*

Audi's production facility for Audi S4's in Ingolstadt (Germany) works around the clock and produces one Audi S4 per hour. Every car produced is tested and is released only if found performing according to the specifications. Otherwise, defective cars are sent to a special unit in Györ (Hungary) to fix them. Suppose that the probability of a car passing the test (and thus released) is $\frac{3}{4}$, and passing the test is independent at the end of each hour. Note that they have very stringent standards for releasing a luxury car.

(a) Suppose that a car is released from Ingolstadt at 4:00pm today. What is the probability that they will release the next car at 7:00pm today?

> **Solution:**
>
> This means the cars produced from 4 pm to 5 pm, and from 5 pm to 6 pm failed, but the car produced from 6 pm to 7 pm passed. The probability is the product of the three probabilities, which is $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{64}$.

(b) What is the probability that they release from Ingolstadt 2 cars in a 5 hour period? Useful facts: $\binom{5}{1} = \binom{5}{4} = 5$; $\binom{5}{2} = \binom{5}{3} = 10$, $4^5 = 1024$.

> **Solution:**
>
> This is a Binomial$(5, 3/4)$ random variable. There are 5 cars built, and 2 must pass inspection and be released. The answer is
>
> $$P = \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 = \frac{45}{512}$$

(c) Suppose every defective car sent to Györ (Hungary) for fixing will cost Audi an additional \$10,000 in transportation and labor. What is the expected cost per 24-hour day that Audi incurs for fixing defective cars?

> **Solution:**
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> The number of defective cars in a 24 hour period is a Binomial$(24, 1/4)$ random variable. The expected number of defective cars is $24 \cdot \frac{1}{4} = 6$. Thus, the expected costs per 24 hour period for Audi is \$60,000.

(d) What is the expected number of hours that Audi requires to manufacture 300 non-defective cars?

> **Solution:**
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> The number of hours X to manufacture one non-defective car is a Geometric$(3/4)$ random variable. Hence, the expected number of hours to manufacture one non-defective car is $\mathbb{E}[X] = \frac{4}{3}$. This gets repeated 300 times, independently of each other. Thus, the expected number of hours to manufacture 300 non-defective cars is $300 \cdot \frac{4}{3} = 400$ hours.

Problem 7 *15 points*

Consider the following game. There are eight balls total of which 4 are red and numbered $1, 2, 3, 4$. The remaining four are blue and also numbered $1, 2, 3, 4$. Three balls are simultaneously pulled out from a hat, and depending on what they are, you might win a prize. Useful facts: $\binom{8}{1} = \binom{8}{7} = 8$; $\binom{8}{2} = \binom{8}{6} = 28$; $\binom{8}{3} = \binom{8}{5} = 56$; $\binom{8}{4} = 70$.

(a) What is the total number of ways that you can draw three balls out of eight?

Solution:

Since the balls are drawn simultaneously, this is a sampling problem that is order independent and without replacement. Therefore, the total number of ways to draw is $\displaystyle\binom{8}{3} = \frac{8!}{3!\,5!} = 56$.

(b) You win a Naruto action figure if all three balls that you draw are the same color. What is the probability of winning a Naruto?

Solution:

There are 2 colors and, for each color, there are $\displaystyle\binom{4}{3} = \frac{4!}{3!\,1!} = 4$ ways to select 3 out of 4 balls. Thus, there are $2 \cdot 4 = 8$ ways to win a Naruto.

$$\mathbb{P}[\{\text{win Naruto}\}] = \frac{\#\text{ ways to win Naruto}}{\#\text{ total ways to draw}} = \frac{8}{56} = \frac{1}{7}.$$

(c) You win a Saint Seiya action figure if two of the balls have the same number. What is the probability that you win a Saint Seiya action figure?

Solution:

You can either win with balls that match at numbers 1, 2, 3 or 4. For each of these matches, there are 6 other choices for the third ball. Thus there are $4 \times 6 = 24$ outcomes for which you get a pair of matching numbers. Since all outcomes are equally likely,

$$\mathbb{P}[\{\text{win Seiya}\}] = \frac{\#\text{ ways to win Seiya}}{\#\text{ total ways to draw}} = \frac{24}{56} = \frac{3}{7}.$$

(d) What is the probability that you win either prize?

Solution:

With a little thinking, you realize that you can't win both prizes. You can't have matching numbers of the same color. Since the outcomes are disjoint,

$$\mathbb{P}[\{\text{win Naruto}\} \cup \{\text{win Saint Seiya}\}] = \frac{1}{7} + \frac{3}{7} = \frac{4}{7}.$$

Problem 8 *4 points*

Two players N (Nadal) and D (Djokovic) are playing a tennis game on clay at the French Open. Player N is serving in this game. Player N wins each point with probability $\frac{2}{3}$ while player D wins with probability $\frac{1}{3}$. Each point may be regarded as an independent trial. Let T denote the event that N and D each win 3 of the first 6 points played. The score is then deuce, meaning tied.

(a) What is $\mathbb{P}(T)$, the probability that deuce was reached? You can leave the answer in terms of combinations and powers. (Hint: need 3 wins for each player in the first six points played.) Useful facts: $\binom{6}{1} = 6; \binom{6}{2} = \binom{6}{4} = 15; \binom{6}{3} = 20;\ 2^6 = 64; 3^6 = 729.$

> **Solution:**
>
> This is a binomial question, where, in 6 trials, you get 3 wins for player A and 3 wins for player B. Hence, $\mathbb{P}(T) = \frac{6*5*4}{3*2*1}(\frac{2}{3})^3(\frac{1}{3})^3 = \frac{160}{729}$.

(b) Once the score reaches deuce, a player must win two more points than the opponent in order to win the game. Given that the score is deuce (Event T occurs), what is the (conditional) probability that N wins the next two points (and hence the game)?

> **Solution:**
>
> It does not matter how the players got to deuce. Since each point is independently played, the probability that N wins the next two points is $(\frac{2}{3})^2 = \frac{4}{9}$.

(c) Let X denote the number of points played after the players reach the first deuce, (event T), until someone wins the game by being ahead 2 points. What is the expected value of X given T, the event that deuce was reached after 6 points?.

> **Solution:**
>
> Note that it takes a multiple of 2 points to end the game after deuce (win by 2). After two points, there are four outcomes: NN, ND, DN, DD. The probability of NN is $\frac{4}{9}$, as computed above, and $\mathbb{P}[\{DD\}] = \frac{1}{9}$. The game ends in outcomes $\{NN, DD\}$ with probability $\frac{5}{9}$, or else it returns to deuce. Thus, the number of pairs of points to play after deuce is reached is a Geometric($\frac{5}{9}$) random variable.
>
> Given this, the expected number of pairs of points to play until the game ends is $\mathbb{E}[X/2] = \frac{9}{5}$. Thus, $\mathbb{E}[X] = \frac{18}{5}$, a little less than 4 points.