Exam 1 Solutions

Problem 1 *8 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$. The sample space is denoted by Ω.

(a) If $B_1, \dots, B_n$ are mutually exclusive, then $\mathbb{P}[A] \leq \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i]$

Solution:
False. To make a partition, consider the collection $B_1, \dots, B_n, C$ where $C = \left(\bigcup_{i=1}^{n} B_i\right)^{\mathsf{c}}$ captures the missing elements. From the Law of Total Probability, we have

$$\mathbb{P}[A] = \mathbb{P}[A|C]\,\mathbb{P}[C] + \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i] \geq \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i] \ ,$$

meaning that the inequality show go in the other direction. (The initial version of the solutions incorrectly said this was "True" due to a typo. However, in regrading student work, we awarded full points to students who wrote "False" but did not take away points from students who wrote "True" and were initially given full points. In effect, that means everyone who answered this problem got full credit.)

(b) $(A \cap B)^{\mathsf{c}}$, $(A^{\mathsf{c}} \cap B)^{\mathsf{c}}$, $(A \cap B^{\mathsf{c}})^{\mathsf{c}}$, and $(A^{\mathsf{c}} \cap B^{\mathsf{c}})^{\mathsf{c}}$ forms a partition.

Solution:

False. Note that $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$ and $(A^{\mathsf{c}} \cap B)^{\mathsf{c}} = A \cup B^{\mathsf{c}}$, meaning that these sets overlap and are not mutually exclusive. Thus, the collection of sets cannot be a partition.

(c) Suppose $\mathbb{P}[A|B] > \mathbb{P}[A|B^{\mathsf{c}}]$ with $\mathbb{P}[B] = 1/2$.
Then, given A occurs, we have that B is more likely to occur than B^{c}.

Solution:
True. First, note that we know $\mathbb{P}[B^{\mathsf{c}}] = 1 - \mathbb{P}[B] = 1 - \frac{1}{2} = \frac{1}{2}$. Using Bayes' rule combined with the Law of Total Probability, we have

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A|B]\,\mathbb{P}[B]}{\mathbb{P}[A|B]\,\mathbb{P}[B] + \mathbb{P}[A|B^{\mathsf{c}}]\,\mathbb{P}[B^{\mathsf{c}}]} = \frac{\mathbb{P}[A|B] \cdot \frac{1}{2}}{\mathbb{P}[A|B] \cdot \frac{1}{2} + \mathbb{P}[A|B^{\mathsf{c}}] \cdot \frac{1}{2}} = \frac{\mathbb{P}[A|B]}{\mathbb{P}[A|B] + \mathbb{P}[A|B^{\mathsf{c}}]} > \frac{1}{2}$$

where the last step uses the fact that $\mathbb{P}[A|B] > \mathbb{P}[A|B^{\mathsf{c}}]$. By the complement property, we know that $\mathbb{P}[B^{\mathsf{c}}|A] = 1 - \mathbb{P}[B|A] < \frac{1}{2}$, so, given A occurs, B is more likely than B^{c}.

(d) If A and B are conditionally independent given C, then $\mathbb{P}[A \cap B|C] = \mathbb{P}[A \cap B|C^{\mathsf{c}}]$.

Solution:

False. For a counterexample, assume $C = A \cap B$. Then $\mathbb{P}[A \cap B | C] = 1$ and $\mathbb{P}[A \cap B | C^{\mathsf{c}}] = 0$.

Problem 2 *8 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) $\mathbb{E}[X^2] = \big(\mathbb{E}\big[|X|\big]\big)^2$. (Recall that $|X|$ is the absolute value of X.)

Solution:
False. Let X have PMF $P_X(x) = \begin{cases} \frac{1}{2} & x = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$
Then, $\mathbb{E}\big[|X|\big] = |0| \cdot \frac{1}{2} + |2| \cdot \frac{1}{2} = 1$ and $\big(\mathbb{E}\big[|X|\big]\big)^2 = 1^2 = 1$. However, $\mathbb{E}[X^2] = 0^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = 2$.

(b) Let X be Binomial(n, p) with $p > 1/2$. Let $B = \{0, 1, 2, \dots, m\}$ with $m < n$. Then, the conditional PMF $P_{X|B}(x)$ corresponds to Binomial(m, p). (Note that $\{X \in B\}$ corresponds to the event that you win at most m games out of n.)

Solution:

False. Let $n = 2$ and $p = 1/2$. The PMF is

$$P_X(x) = \begin{cases} \binom{2}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{2-x} & x = 0, 1, 2, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \frac{1}{4} & x = 0, 2, \\ \frac{1}{2} & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $m = 1$ and note that $\mathbb{P}[X \in B] = P_X(0) + P_X(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. The corresponding conditional PMF is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{\mathbb{P}[X \in B]} & x \in B \\ 0 & x \notin B \end{cases} = \begin{cases} \frac{1/4}{3/4} & x = 0, \\ \frac{1/2}{3/4} & x = 1, \\ 0 & \text{otherwise.} \end{cases} \quad \begin{cases} \frac{1}{3} & x = 0, \\ \frac{2}{3} & x = 1, \\ 0 & \text{otherwise,} \end{cases}$$

which is Binomial$(1, 1/3)$ not Binomial$(1, 1/2)$.

(c) Let $A = \big\{x \in R_X : x < \mathbb{E}[X]\big\}$. Then, $\mathbb{E}[X|A] < \mathbb{E}[X]$.

Solution:

True. $\mathbb{E}[X|A] = \sum_{x \in A} x \, P_{X|A}(x) < \sum_{x \in A} \mathbb{E}[X] \, P_{X|A}(x) = \mathbb{E}[X]$ where the inequality uses the fact that all elements of A are less that $\mathbb{E}[X]$.

(d) If $\mathbb{E}[(X-1)^2] = \mathbb{E}[X^2] + 1$, then $\mathbb{E}[X] = 0$.

Solution:

True. Using the linearity of expectation, $\mathbb{E}[(X-1)^2] = \mathbb{E}[X^2 - 2X + 1] = \mathbb{E}[X^2] - 2\mathbb{E}[X] + 1$. If this is equal to $\mathbb{E}[X^2] + 1$, then $-2\mathbb{E}[X] = 0$, which implies $\mathbb{E}[X] = 0$.

Problem 3 *16 points*

Complete the following quick calculations. **For full credit, you must work out a numerical answer for each requested quantity in this problem.**

(a) Let X be Binomial$(3, \frac{1}{3})$. Compute $\mathbb{P}[X > 1]$ and $\mathbb{P}[2^X < 4]$.

Solution:

$$\mathbb{P}[X > 1] = P_X(2) + P_X(3) = \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 + \binom{3}{3}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 = 3 \cdot \frac{1 \cdot 2}{3^3} + \frac{1 \cdot 1}{3^3} = \frac{7}{27}$$

$$\mathbb{P}[2^X < 4] = \mathbb{P}[X < 2] = 1 - \mathbb{P}[X > 1] = 1 - \frac{7}{27} = \frac{20}{27}$$

(b) Let X be Geometric$(\frac{1}{2})$. Calculate $\mathsf{Var}[2X+1]$ and $\mathbb{E}[X^2 - 4]$.

Solution:

For Geometric(p) RVs, $\mathbb{E}[X] = \frac{1}{p}$ and $\mathsf{Var}[X] = \frac{1-p}{p^2}$. Using the variance of a linear function, we have $\mathsf{Var}[2X+1] = 2^2\,\mathsf{Var}[X] = 4\,\mathsf{Var}[X] = 4 \cdot \frac{1/2}{(1/2)^2} = 4 \cdot 2 = 8$. Using the alternate variance formula, we have $\mathbb{E}[X^2] = \mathsf{Var}[X] + (\mathbb{E}[X])^2 = 2 + (\frac{1}{1/2})^2 = 2 + 2^2 = 6$. Using the linearity of expectation, $\mathbb{E}[X^2 - 4] = \mathbb{E}[X^2] - 4 = 6 - 4 = 2$.

(c) Let X be a Poisson(1) and $A = \{1, 3\}$. Calculate $\mathbb{P}_{X|A}(x)$ and $\mathbb{E}[10^X|A]$.

Solution:

We only need to evaluate $P_X(x)$ for $x \in A$ to determine $\mathbb{P}_{X|A}(x)$. For a Poisson(1) RV, we have $P_X(1) = \frac{1^1}{1!}e^{-1} = e^{-1}$ and $P_X(3) = \frac{1^3}{3!}e^{-1} = \frac{1}{6}e^{-1}$. Therefore, $\mathbb{P}[X \in A] = P_X(1) + P_X(3) = \frac{7}{6}e^{-1}$ and

$$P_{X|A}(x) = \begin{cases} \dfrac{P_X(x)}{\mathbb{P}[X \in A]} & x \in A \\ 0 & x \notin A \end{cases} = \begin{cases} \frac{e^{-1}}{\frac{7}{6}e^{-1}} & x = 1 \\ \frac{\frac{1}{6}e^{-1}}{\frac{7}{6}e^{-1}} & x = 3 \\ 0 & x \notin A \end{cases} = \begin{cases} \frac{6}{7} & x = 1 \\ \frac{1}{7} & x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[10^X|A] = \sum_{x \in A} 10^x P_{X|A}(x) = 10^1 P_{X|A}(1) + 10^3 P_{X|A}(3) = 10 \cdot \frac{6}{7} + 1000 \cdot \frac{1}{7} = \frac{1060}{7}$$

(d) Let A, B, C be events with $\mathbb{P}[A|C] = \mathbb{P}[B|C] = \mathbb{P}[B|C^{\mathsf{c}}] = \frac{1}{4}$ and $\mathbb{P}[C] = \mathbb{P}[A|C^{\mathsf{c}}] = \frac{1}{2}$. Additionally, let A and B be independent. Calculate $\mathbb{P}[A \cap B]$ and $\mathbb{P}[A \cup B]$.

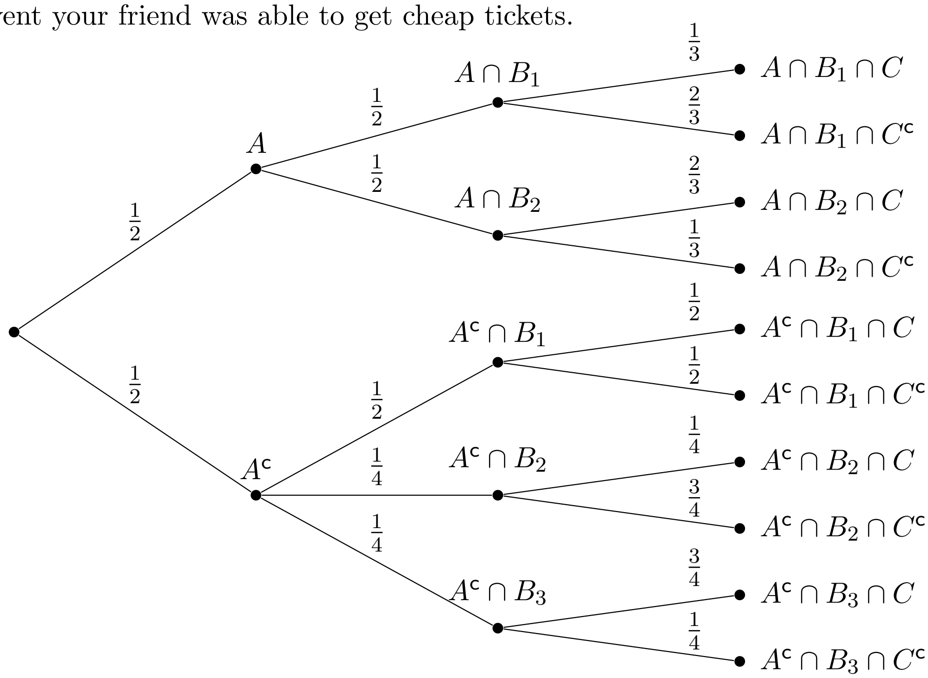
Solution:

Since A and B are independent, we know that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$. We can use the Law of Total Probability to calculate these probabilities.

$$
\begin{aligned}
\mathbb{P}[A] &= \mathbb{P}[A|C]\,\mathbb{P}[C] + \mathbb{P}[A|C^{\mathsf{c}}]\,\mathbb{P}[C^{\mathsf{c}}] = \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{8} \\
\mathbb{P}[B] &= \mathbb{P}[B|C]\,\mathbb{P}[C] + \mathbb{P}[B|C^{\mathsf{c}}]\,\mathbb{P}[C^{\mathsf{c}}] = \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{4} \\
\mathbb{P}[A\cap B] &= \mathbb{P}[A]\,\mathbb{P}[B] = \frac{3}{8}\cdot\frac{1}{4} = \frac{3}{32} \\
\mathbb{P}[A\cup B] &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A\cap B] = \frac{3}{8} + \frac{1}{4} - \frac{3}{32} = \frac{12+8-3}{32} = \frac{17}{32}
\end{aligned}
$$

Problem 4 *12 points*

Your friend attended a concert. Let A be the event it was held in the usual auditorium. Let B_1 be the event that it was your friend's favorite band, B_2 their second favorite, and B_3 their third favorite. Let C be the event your friend was able to get cheap tickets.



(a) Are the events A and B_1 independent? Justify your answer with calculations.

Solution:

We need to check if $\mathbb{P}[A\cap B_1] = \mathbb{P}[A]\,\mathbb{P}[B_1]$. From the tree, $\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[A\cap B_1] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, and $\mathbb{P}[B] = \mathbb{P}[A\cap B_1] + \mathbb{P}[A^{\mathsf{c}}\cap B_1] = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$. Since $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, we see that, yes, A and B_1 are independent.

(b) Given it was their second favorite, what is the probability it was the usual auditorium **and** cheap tickets?

Solution:

$$
\mathbb{P}[A\cap C|B_2] = \frac{\mathbb{P}[A\cap B_2\cap C]}{\mathbb{P}[B_2]} = \frac{\mathbb{P}[A\cap B_2\cap C]}{\mathbb{P}[A\cap B_2] + \mathbb{P}[A^{\mathsf{c}}\cap B_2]} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4}} = \frac{\frac{2}{12}}{\frac{3}{8}} = \frac{4}{9}\,.
$$

(c) Given that your friend got cheap tickets, what is the probability it was in the usual auditorium?

Solution:

$$\begin{aligned}
\mathbb{P}[A|C] &= \frac{\mathbb{P}[A\cap C]}{\mathbb{P}[C]} \\
&= \frac{\mathbb{P}[A\cap B_1\cap C]+\mathbb{P}[A\cap B_2\cap C]}{\mathbb{P}[A\cap B_1\cap C]+\mathbb{P}[A\cap B_2\cap C]+\mathbb{P}[A^{\mathsf{c}}\cap B_1\cap C]+\mathbb{P}[A^{\mathsf{c}}\cap B_2\cap C]+\mathbb{P}[A^{\mathsf{c}}\cap B_3\cap C]} \\
&= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}} = \frac{\frac{1}{12}+\frac{2}{12}}{\frac{1}{12}+\frac{2}{12}+\frac{1}{8}+\frac{1}{32}+\frac{3}{32}} = \frac{\frac{3}{12}}{\frac{12}{24}} = \frac{1}{2}\,.
\end{aligned}$$

Problem 5 *16 points*

A pack of 6 widgets contains 3 faulty widgets. You take out 4 widgets. Your formula sheet includes an $\binom{n}{k}$ table so we are expecting exact answers throughout this problem.

(a) How many ways are there to select 4 widgets with exactly 2 faulty?

Solution:

The sampling problem is order independent and without replacement. Therefore, for n items and k choices, we should use the formula $\binom{n}{k}$. The number of ways to select 2 out of 3 faulty and 2 out of 3 non-faulty widgets can be expressed as

$$(\#\text{ ways } 2 \text{ out of } 3 \text{ faulty}) \times (\#\text{ ways } 2 \text{ out of } 3 \text{ non-faulty}) = \binom{3}{2}\times\binom{3}{2} = 3\cdot 3 = 9$$

(b) What is the probability that exactly 2 of the widgets you select are faulty?

Solution:

First, note that there are 6 widgets in total, and the number of ways to select 4 of them is $\binom{6}{4} = 15$. It follows that $\mathbb{P}[\{\text{exactly } 2 \text{ faulty}\}] = \dfrac{\#\text{ ways to select } 2 \text{ faulty}}{\#\text{ total ways to draw}} = \dfrac{9}{15} = \dfrac{3}{5}$.

(c) Let X be a random variable corresponding to the exact number of faulty widgets you select. Determine the PMF $P_X(x)$ and write it as a case-by-case formula.

Solution:

We need to repeat the calculation above for different choices for the number of faulty widgets.

$$\underline{\text{exactly 1:}}\ (\#\text{ ways 1 out of 3 faulty}) \times (\#\text{ ways 3 out of 3 non-faulty}) = \binom{3}{1} \times \binom{3}{3} = 3 \cdot 1 = 3$$
$$\underline{\text{exactly 2:}}\ (\#\text{ ways 2 out of 3 faulty}) \times (\#\text{ ways 2 out of 3 non-faulty}) = \binom{3}{2} \times \binom{3}{2} = 3 \cdot 3 = 9$$
$$\underline{\text{exactly 3:}}\ (\#\text{ ways 3 out of 3 faulty}) \times (\#\text{ ways 1 out of 3 non-faulty}) = \binom{3}{3} \times \binom{3}{1} = 1 \cdot 3 = 3$$

$$P_X(1) = \mathbb{P}[\{\text{exactly 1 faulty}\}] = \frac{\#\text{ ways to select 1 faulty}}{\#\text{ total ways to draw}} = \frac{3}{15} = \frac{1}{5}$$
$$P_X(2) = \mathbb{P}[\{\text{exactly 2 faulty}\}] = \frac{\#\text{ ways to select 2 faulty}}{\#\text{ total ways to draw}} = \frac{9}{15} = \frac{3}{5}$$
$$P_X(3) = \mathbb{P}[\{\text{exactly 3 faulty}\}] = \frac{\#\text{ ways to select 3 faulty}}{\#\text{ total ways to draw}} = \frac{3}{15} = \frac{1}{5}$$
$$P_X(x) = \begin{cases} 1/5 & x = 1, 3, \\ 3/5 & x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(d) What is the average number of faulty widgets you select?

Solution:

By symmetry, we can see $\mathbb{E}[X] = 2$ but this also follows from a direct calculation:

$$\mathbb{E}[X] = \sum_{x \in R_X} x\, P_X(x) = 1 \cdot P_X(1) + 2 \cdot P_X(2) + 3 \cdot P_X(3) = 1 \cdot \frac{1}{5} + 2 \cdot \frac{3}{5} + 3 \cdot \frac{1}{5} = 2$$

Problem 6 *16 points*

The number of patients checking into a hospital per hour X is well modeled as Poisson(3).
We say that an hour is calm if **at most one** patient checks in.

(a) What is the probability that an hour is calm?

Solution:

$$\mathbb{P}[X \leq 1] = P_X(0) + P_X(1) = \frac{3^0}{0!}e^{-3} + \frac{3^1}{1!}e^{-3} = 4e^{-3}$$

(b) Let B be the event that an hour is calm. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula.

Solution:

$\mathbb{P}[B] = \mathbb{P}[X \leq 1] = 4e^{-3}$. We also know $P_X(0) = e^{-3}$ and $P_X(1) = 3e^{-3}$ from above. We restrict and rescale to get the desired conditional PMF:

$$P_{X|B}(x) = \begin{cases} \frac{e^{-3}}{4e^{-3}} & x = 0, \\ \frac{3e^{-3}}{4e^{-3}} & x = 1, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{4} & x = 0, \\ \frac{3}{4} & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Consider now the following scenario. Out of 4 calm hours, we count the total number of patients Y that check in. What kind of random variable is Y? Don't forget the parameters.

Solution:

This is a Binomial$\,(4, 3/4)$ random variable.

(d) What is the probability that at least one patient checks in across 4 calm hours?

Solution:

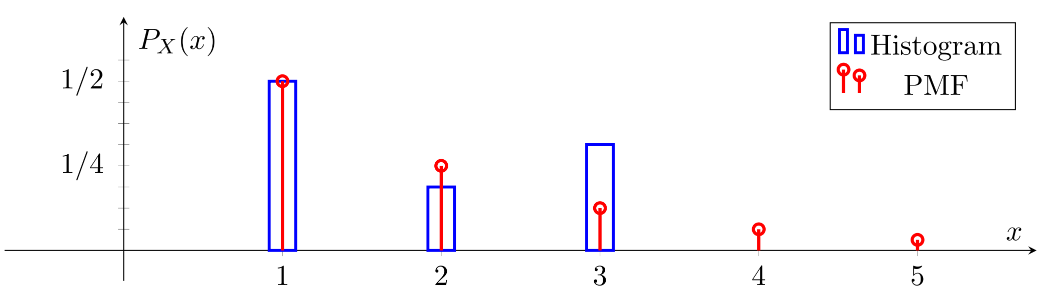
$$\mathbb{P}[Y \geq 1] = 1 - \mathbb{P}[Y = 0] = 1 - \binom{4}{0}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 = 1 - \frac{1}{256} = \frac{255}{256}\,.$$

Problem 7 *16 points*

A cereal company includes prizes in some cereal boxes with probability $\frac{1}{2}$, independently across boxes. Let X be the number of boxes purchased until the first prize. Your friend is working as an intern with this company, and they designed a survey with the following questions and preliminary collected data.

How many boxes until first prize?	# Responses	(a) Histogram Values	(d) Predicted Probabilities
1	8	$\frac{8}{8+3+5} = \frac{1}{2}$	$P_X(1) = \frac{1}{2}$
2	3	$\frac{3}{8+3+5} = \frac{3}{16}$	$P_X(2) = \frac{1}{4}$
3 or more	5	$\frac{5}{8+3+5} = \frac{5}{16}$	$\mathbb{P}[X \geq 3] = \frac{1}{4}$

(a) Using the normalized counts (like we did in the labs), determine the histogram values for this dataset. Fill in your values in both the table above under column (a) Histogram Values and as a bar plot in the figure below. **Your friend has decided you should plot "3 or more" at $x = 3$.**



(b) What kind of random variable is X? Don't forget the parameters. Draw its PMF as a stem plot on the figure above for all x values on the x-axis.

Solution:

X is Geometric$(1/2)$, which has PMF $P_X(x) = \begin{cases} (\frac{1}{2})^x & x = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$.

(c) Your friend believes that, if they collect more data with the existing survey questions, the histogram and PMF will eventually agree. Do you agree, yes or no? Support your answer with probability concepts.

Solution:

The correct answer was no. Of course, if we collect additional data, the estimated probabilities will converge to the true probabilities. However, your friend is mistakenly plotting "3 or more" at $x = 3$, whereas the PMF will only plot the probability of "exactly 3" at $x = 3$. (If you mentioned that collecting more data should lead to better probability estimates, you got partial credit, but full credit was reserved for correctly identifying the reason the histogram and PMF look different, even with the small dataset we have now.)

(d) Using the PMF for the random variable X, determine the predicted values for the survey questions responses, assuming your friend collects much more data. Write your answers in column (d) Predicted Probabilities in the table above.

Solution:

We can use the PMF to predict the values, even for the "3 or more" question. Specifically, we have that

$$\mathbb{P}[X = 1] = P_X(1) = \frac{1}{2}$$
$$\mathbb{P}[X = 2] = P_X(2) = \frac{1}{4}$$
$$\mathbb{P}[X \geq 3] = 1 - \mathbb{P}[X < 3] = 1 - P_X(1) - P_X(2) = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

and we can see that the small dataset we have is already hovering around these values.

Problem 8 *8 points*

Your company has created a collection of N machine learning algorithms for classifying images as either "cat" or "dog." Your dataset contains more dog images than cat images, and you may assume the probability of getting a dog image is $\frac{2}{3}$ and, otherwise, you get a cat image. Each algorithm classifies a cat image correctly as "cat" with probability p and a dog image correctly as "dog" with probability q. The decisions of each algorithm can be modeled as independent of one another. Your insight is that, to improve the overall performance, you can run all N algorithms on an image, and use the majority vote as the output. For instance, if $N = 3$ and the algorithms output the guesses "cat," "dog," "cat," you will guess this is a cat image. You may assume throughout the problem that N is odd.

(a) What is the probability your majority-voting algorithm is correct? (It is OK to leave your answer in terms of summations.)

Solution:

Let X be the total number of correct guesses out of N for a cat image. We need to start by recognizing that X is Binomial(N, p). Similarly, let Y be the total number of correct guesses out of N for a dog image. We have that Y is Binomial(N, q). Since N is odd, a majority of the guesses will be correct if at least $\frac{N+1}{2}$ guesses are correct. Thus, we have

$$\mathbb{P}[\{\text{majority vote correct}\}|\{\text{cat image}\}] = \mathbb{P}\left[X \geq \frac{N+1}{2}\right] = \sum_{i=\frac{N+1}{2}} \binom{N}{i} p^i (1-p)^{N-i} ,$$

$$\mathbb{P}[\{\text{majority vote correct}\}|\{\text{dog image}\}] = \mathbb{P}\left[Y \geq \frac{N+1}{2}\right] = \sum_{i=\frac{N+1}{2}} \binom{N}{i} q^i (1-q)^{N-i} .$$

Now, we can use the Law of Total Probability to put these together,

$$\begin{aligned}\mathbb{P}[\{\text{majority vote correct}\}] &= \mathbb{P}[\{\text{majority vote correct}\}|\{\text{cat image}\}]\mathbb{P}[\{\text{cat image}\}] \\ &\quad + \mathbb{P}[\{\text{majority vote correct}\}|\{\text{dog image}\}]\mathbb{P}[\{\text{dog image}\}] \\ &= \frac{1}{3} \sum_{i=\frac{N+1}{2}} \binom{N}{i} p^i (1-p)^{N-i} + \frac{2}{3} \sum_{i=\frac{N+1}{2}} \binom{N}{i} q^i (1-q)^{N-i}\end{aligned}$$

(b) Given that your majority-voting algorithm was correct, what is the probability it was actually a cat image for $N = 3$, $p = 2/3$, $q = 1/2$? (For full credit, please provide a numerical answer with all terms simplified.)

Solution:

First, we calculate some of the terms above for the given parameters:

$$\begin{aligned}\mathbb{P}[\{\text{majority vote correct}\}|\{\text{cat image}\}] &= \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 + \binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{12}{27} + \frac{8}{27} = \frac{20}{27} \\ \mathbb{P}[\{\text{majority vote correct}\}|\{\text{dog image}\}] &= \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 + \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{3}{8} + \frac{1}{8} = \frac{1}{2} \\ \mathbb{P}[\{\text{majority vote correct}\}] &= \frac{1}{3} \cdot \frac{20}{27} + \frac{2}{3} \cdot \frac{1}{2} = \frac{20}{81} + \frac{1}{3} = \frac{47}{81} .\end{aligned}$$

Now, using Bayes' Rule,

$$\begin{aligned}\mathbb{P}[\{\text{cat image}\}|\{\text{majority vote correct}\}] &= \frac{\mathbb{P}[\{\text{majority vote correct}\}|\{\text{cat image}\}]\,\mathbb{P}[\{\text{cat image}\}]}{\mathbb{P}[\{\text{majority vote correct}\}]} \\ &= \frac{\frac{20}{27} \cdot \frac{1}{3}}{\frac{47}{81}} = \frac{20}{47} .\end{aligned}$$