Exam 1 Solutions

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$. The sample space is denoted by Ω.

(a) If $A \cup B = \Omega$, then $\mathbb{P}[A] + \mathbb{P}[B] = 1$.

> **Solution:**
>
> **False.** Any situation with $\mathbb{P}[A \cap B] > 0$ is a counterexample.

(b) If $\mathbb{P}[A \mid B] < \mathbb{P}[A]$, then $\mathbb{P}[A \cap B] > \mathbb{P}[B]$.

> **Solution:**
>
> **False.** Since $A \cap B \subset B$, it is never true that $\mathbb{P}[A \cap B] > \mathbb{P}[B]$.

(c) $\mathbb{P}[A \cap B^c] = \mathbb{P}[A] - \mathbb{P}[A \cap B]$

> **Solution:**
>
> **True.** By adding $\mathbb{P}[A \cap B]$ to both sides, we get $\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^c]$, which holds by the Law of Total Probability.

(d) $\mathbb{P}[A^c \cap B^c] = 1 - \mathbb{P}[A] - \mathbb{P}[B]$.

> **Solution:**
>
> **False.** Though this looks like it might be derived by applying De Morgan's law, any situation with $\mathbb{P}[A \cap B] > 0$ is a counterexample.

(e) $\mathbb{P}[A] + \mathbb{P}[A^c] + \mathbb{P}[B] > 1$.

> **Solution:**
>
> **True.** The first two terms always add to 1. The third term being positive was given in the problem statement.

Problem 2 *8 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $\mathbb{E}[(X+3)^2] = \mathbb{E}[X^2] + 9$, then $\mathbb{E}[X] = 0$.

Solution:

True. $\mathbb{E}[(X+3)^2] = \mathbb{E}[X^2+6X+9] = \mathbb{E}[X^2] + 6\,\mathbb{E}[X] + 9$, where the second step uses linearity of expectation. Thus $6\,\mathbb{E}[X] = 0$.

(b) For some values of the constant c, $\mathsf{Var}[cX] < \mathsf{Var}[X]$.

Solution:

True. $\mathsf{Var}[cX] = c^2\,\mathsf{Var}[X]$. When $|c| < 1$, $c^2 < 1$.

(c) If for some constant c we have $\mathbb{P}[X < c] > 1/2$, then $\mathbb{E}[X] < c$.

Solution:

False. Take $X \sim \text{Bernoulli}(1/3)$ (so that $\mathbb{E}[X] = 1/3$) and consider $c = 1/4$ (so that $\mathbb{P}[X < c] = 2/3$).

(d) If $F_X(a) > F_X(b)$, then $a > b$.

Solution:

True. This follows from CDFs being nondecreasing.

Problem 3 Complete the following quick calculations. *16 points*

(a) Consider independent events A, B, and C satisfying $\mathbb{P}[A] = 1/3$, $\mathbb{P}[B] = 1/2$, $\mathbb{P}[C] = 2/3$. Calculate $\mathbb{P}[A \cap C^c]$ and $\mathbb{P}[A \cup (B \cap C)]$.

Solution:

Since A and C^c are independent,

$$\mathbb{P}[A \cap C^c] = \mathbb{P}[A]\,\mathbb{P}[C^c] = \frac{1}{3} \cdot \left(1 - \frac{2}{3}\right) = \frac{1}{9}.$$

Using independence and the inclusion–exclusion formula,

$$\begin{aligned}\mathbb{P}[A \cup (B \cap C)] &= \mathbb{P}[A] + \mathbb{P}[B \cap C] - \mathbb{P}[A \cap B \cap C] \\ &= \mathbb{P}[A] + \mathbb{P}[B]\,\mathbb{P}[C] - \mathbb{P}[A]\,\mathbb{P}[B]\,\mathbb{P}[C] \\ &= \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3} - \frac{1}{9} = \frac{5}{9}.\end{aligned}$$

(b) Let A and B be events with $\mathbb{P}[A \mid B] = 1/3$, $\mathbb{P}[A \mid B^c] = 1/6$ and $\mathbb{P}[B^c] = 3/4$. Calculate $\mathbb{P}[A \cap B]$ and $\mathbb{P}[B \mid A]$.

Solution:

Since $\mathbb{P}[B] = 1 - \mathbb{P}[B^c] = 1/4$, the multiplication rule gives

$$\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B]\,\mathbb{P}[B] = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}.$$

We similarly compute

$$\mathbb{P}[A \cap B^c] = \mathbb{P}[A \mid B^c]\,\mathbb{P}[B^c] = \frac{1}{6} \cdot \frac{3}{4} = \frac{1}{8}.$$

Then

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]} = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^c]} = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{8}} = \frac{2}{5}.$$

(c) Let X be Binomial$(4, \frac{1}{3})$. Calculate $\mathbb{P}[X \geq 1]$ and $\mathbb{E}[4X - 2]$.

Solution:

$$\mathbb{P}[X \geq 1] = 1 - P_X(0) = 1 - \binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 = 1 - \frac{16}{81} = \frac{65}{81}$$

$$\mathbb{E}[4X - 2] = 4\,\mathbb{E}[X] - 2 = 4 \cdot \frac{4}{3} - 2 = \frac{10}{3}$$

(d) Let X be Discrete Uniform$(-1, 2)$. Calculate $\mathbb{E}[X^2]$ and $\mathbb{E}[X^2 \mid X < 2]$.

Solution:

X takes values in $\{-1, 0, 1, 2\}$, each with probability $1/4$. Thus

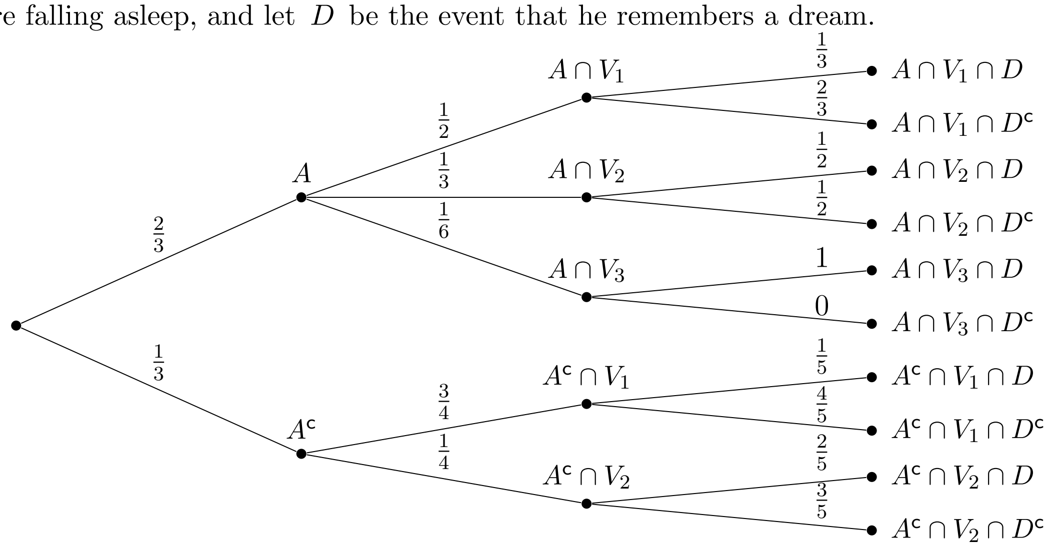
$$\mathbb{E}[X^2] = (-1)^2\frac{1}{4} + (0)^2\frac{1}{4} + (1)^2\frac{1}{4} + (2)^2\frac{1}{4} = \frac{1+0+1+4}{4} = \frac{3}{2}.$$

Conditioned on $\{X < 2\}$, X takes values in $\{-1, 0, 1\}$, each with probability $1/3$. Thus

$$\mathbb{E}[X^2 \mid X < 2] = (-1)^2\frac{1}{3} + (0)^2\frac{1}{3} + (1)^2\frac{1}{3} = \frac{2}{3}.$$

Problem 4 *16 points*

Vincent uses the tree diagram below for a model of whether he wakes up remembering a dream. Let A be the event that he drinks absinthe before going to bed, let V_i be the event that he plays i games of Wordle in bed before falling asleep, and let D be the event that he remembers a dream.



(a) What is the probability that Vincent plays two games of Wordle before falling asleep?

Solution:

We wish to find the probability of V_2:

$$\mathbb{P}[V_2] = \mathbb{P}[V_2|A]\,\mathbb{P}[A] + \mathbb{P}[V_2|A^c]\,\mathbb{P}[A^c] = \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{2}{9} + \frac{1}{12} = \frac{8+3}{36} = \frac{11}{36}$$

(b) Given that Vincent drank absinthe before going to bed, what is the probability that he remembers a dream?

Solution:

We wish to find $\mathbb{P}[D \mid A]$. Since we are conditioning on A, we can consider only the descendants of the A node in the tree. In that part of the tree, D is partitioned as $D = (V_1 \cap D) \cup (V_2 \cap D) \cup (V_3 \cap D)$. Thus

$$\begin{aligned}
\mathbb{P}[D \mid A] &= \mathbb{P}[V_1 \cap D \mid A] + \mathbb{P}[V_2 \cap D \mid A] + \mathbb{P}[V_3 \cap D \mid A] \\
&= \mathbb{P}[D \mid (A \cap V_1)]\,\mathbb{P}[V_1 \mid A] + \mathbb{P}[D \mid (A \cap V_2)]\,\mathbb{P}[V_2 \mid A] + \mathbb{P}[D \mid (A \cap V_3)]\,\mathbb{P}[V_3 \mid A] \\
&= \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{6} = \frac{1}{2}.
\end{aligned}$$

(c) Given that Vincent remembers a dream, what is the probability that he drank absinthe before going to bed?

Solution:

We wish to find $\mathbb{P}[A \mid D] = \mathbb{P}[A \cap D]/\mathbb{P}[D]$. By using the computation from (b) and label from the first level of the tree,

$$\mathbb{P}[A \cap D] = \mathbb{P}[D \mid A]\,\mathbb{P}[A] = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}.$$

For $\mathbb{P}[D]$, we first make a computation analogous to part (b):

$$\begin{aligned}
\mathbb{P}[D \mid A^c] &= \mathbb{P}[V_1 \cap D \mid A^c] + \mathbb{P}[V_2 \cap D \mid A^c] \\
&= \mathbb{P}[D \mid (A^c \cap V_1)]\,\mathbb{P}[V_1 \mid A^c] + \mathbb{P}[D \mid (A^c \cap V_2)]\,\mathbb{P}[V_2 \mid A^c] \\
&= \frac{1}{5}\cdot\frac{3}{4} + \frac{2}{5}\cdot\frac{1}{4} = \frac{1}{4}.
\end{aligned}$$

Then by the Law of Total Probability,

$$\mathbb{P}[D] = \mathbb{P}[D \mid A]\,\mathbb{P}[A] + \mathbb{P}[D \mid A^c]\,\mathbb{P}[A^c] = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{5}{12}.$$

So our final answer is

$$\mathbb{P}[A \mid D] = \frac{1/3}{5/12} = \frac{4}{5}.$$

(d) Given that Vincent played two games of Wordle and remembers a dream, what is the probability that he drank absinthe?

Solution:

We wish to find $\mathbb{P}[A \,|\, (V_2 \cap D)]$, and this requires just two leaf node probabilities:

$$\mathbb{P}[A \cap V_2 \cap D] = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{9}$$
$$\mathbb{P}[A^c \cap V_2 \cap D] = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{2}{5} = \frac{1}{30}$$

because

$$\mathbb{P}[A \,|\, (V_2 \cap D)] = \frac{\mathbb{P}[A \cap V_2 \cap D]}{\mathbb{P}[V_2 \cap D]} = \frac{\mathbb{P}[A \cap V_2 \cap D]}{\mathbb{P}[A \cap V_2 \cap D] + \mathbb{P}[A^c \cap V_2 \cap D]} = \frac{1/9}{1/9 + 1/30} = \frac{10}{13}.$$

Problem 5 *12 points*

Each part of this problem uses a well-shuffled standard 52-card deck: four suits (spades, hearts, diamonds, and clubs), 13 cards of each suit (numbers from 2 to 10, Jack, Queen, King, and Ace). Remember that expressions with factorials and binomial coefficients are fine in your final answers.

(a) What is the probability that a 3-card hand is all spades?

Solution:

By considering drawing the cards in sequence, we get

$$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50}.$$

(b) What is the probability that a 6-card hand has 4 cards of one suit and 2 cards of a second suit?

Solution:

The total number of 6-card hands is $\binom{52}{6}$. To count the number of hands satisfying the criterion: there are 4 suits and $\binom{13}{4}$ possible sets of cards within the first suit; and there are 3 remaining suits and $\binom{13}{2}$ possible sets of cards within the second suit. So the probability is

$$\frac{(4)(3)\binom{13}{4}\binom{13}{2}}{\binom{52}{6}}.$$

(c) Given that a 6-card hand contains at least two suits, what is the probability that the 6 cards are 4 of one suit and 2 of a second suit?

Solution:

We have already counted the number of hands that have 4 of one suit and 2 of another. For the conditional probability, we must reduce the universe of possible hands to exclude those that contain only one suit. Since there are 4 choices of suit and $\binom{13}{6}$ possible sets of cards within that suit, the probability of interest is

$$\frac{(4)(3)\binom{13}{4}\binom{13}{2}}{\binom{52}{6} - 4\binom{13}{6}}.$$

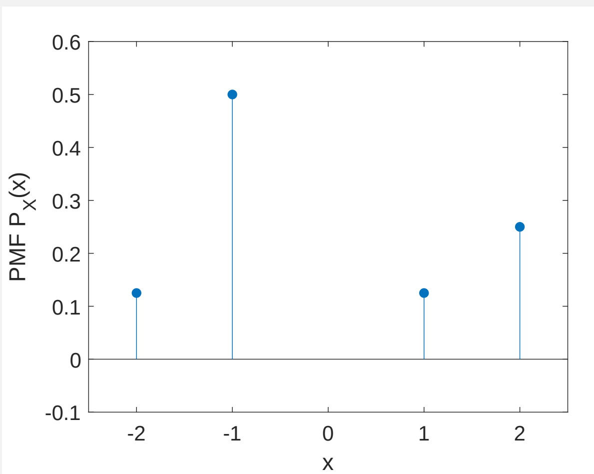
Problem 6 *16 points*

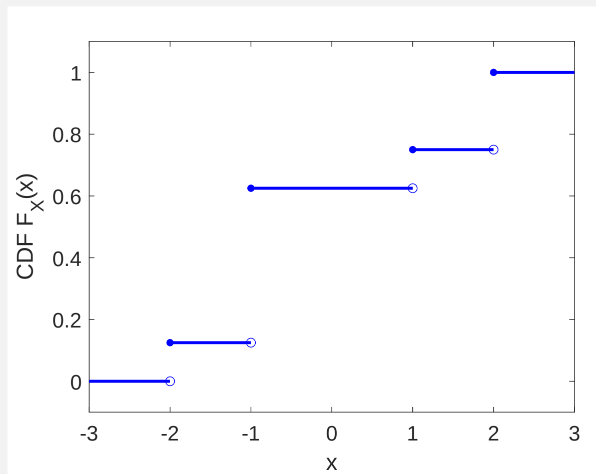
Let X have the probability mass function

$$P_X(x) = \begin{cases} 1/8, & x = -2; \\ 1/2, & x = -1; \\ 1/8, & x = 1; \\ 1/4, & x = 2; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Make clearly labeled sketches of the PMF $P_X(x)$ and CDF $F_X(x)$.

Solution:





(b) Calculate $\mathbb{E}[X]$ and $\mathsf{Var}[X]$.

Solution:

$$\begin{aligned} \mathbb{E}[X] &= (-2)\frac{1}{8} + (-1)\frac{1}{2} + (1)\frac{1}{8} + (2)\frac{1}{4} = \frac{-2-4+1+4}{8} = -\frac{1}{8} \\ \mathbb{E}[X^2] &= (-2)^2\frac{1}{8} + (-1)^2\frac{1}{2} + (1)^2\frac{1}{8} + (2)^2\frac{1}{4} = \frac{4+4+1+8}{8} = \frac{17}{8} \\ \mathsf{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{17}{8} - \left(-\frac{1}{8}\right)^2 = \frac{135}{64} \end{aligned}$$

(c) Let B be the event that $X < 2$. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula.

Solution:

$\mathbb{P}[B] = 1 - P_X(2) = 3/4$. We restrict and divide by $3/4$ to get the desired conditional PMF:

$$P_{X|B}(x) = \begin{cases} \frac{1/8}{3/4}, & x = -2; \\ \frac{1/2}{3/4}, & x = -1; \\ \frac{1/8}{3/4}, & x = 1; \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 1/6, & x = -2; \\ 2/3, & x = -1; \\ 1/6, & x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Compute $\mathbb{E}\left[\frac{1}{X} \mid B\right]$.

Solution:

$$\begin{aligned} \mathbb{E}\left[\frac{1}{X} \mid B\right] &= \sum_{x \in \{-2,-1,1\}} \frac{1}{x} P_{X|B}(x) \\ &= -\frac{1}{2} P_{X|B}(-2) + (-1) P_{X|B}(-1) + 1 P_{X|B}(1) \\ &= -\frac{1}{2} \cdot \frac{1}{6} + (-1) \cdot \frac{2}{3} + 1 \cdot \frac{1}{6} = \frac{-1 - 8 + 2}{12} = -\frac{7}{12} \end{aligned}$$

Problem 7 *16 points*

The number of spam text messages you receive each day has the Poisson(2) distribution, and the number of these messages is independent from day to day.

(a) Let X be the number of days in a week (7 days) that you receive no spam text messages. What is $\mathsf{Var}[X]$?

Solution:

X is a binomial random variable with $n = 7$ trials. The probability parameter p is the probability of receiving no spam text messages: $p = e^{-2}$. Using a standard formula for binomial random variables, $\mathsf{Var}[X] = np(1-p) = 7e^{-2}(1 - e^{-2})$.

(b) What is the probability that in one week, there are exactly 4 days on which you receive exactly 3 spam text messages?

Solution:

Let q be the probability of receiving exactly 3 spam text messages on some day. Then based on evaluating the Poisson PMF, $q = e^{-2}2^3/3! = \frac{4}{3}e^{-2}$. The probability we want is a binomial probability for 4 successes in 7 trial with parameter q:

$$\binom{7}{4}\left(\frac{4}{3}e^{-2}\right)^4\left(1 - \frac{4}{3}e^{-2}\right)^3.$$

(c) Let Y be the number of days until the first on which you receive 2 or more spam text messages. What is $\mathbb{E}[Y]$?

Solution:

The probability of receiving 2 or more spam text messages is

$$r = 1 - e^{-2}\frac{2^0}{0!} - e^{-2}\frac{2^1}{1!} = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}.$$

Y is a geometric (r) random variable, so

$$\mathbb{E}[Y] = \frac{1}{r} = \frac{1}{1-3e^{-2}}.$$

(d) Given that you have received at most 3 spam text messages in a day, what is the conditional PMF of the number of spam text messages received?

Solution:

Let U be the number of spam text messages received, and let $B = \{0,1,2,3\}$. We find $P_{U|B}(u)$ by restricting and renormalizing the Poisson(2) distribution.

$$\mathbb{P}[U \in B] = \sum_{u=0}^{3} e^{-2}\frac{2^u}{u!} = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = e^{-2}\left(1+2+2+\frac{8}{6}\right) = e^{-2}\frac{19}{3}$$

The desired conditional PMF is given by the values $(1,2,2,\frac{4}{3})$ normalized by $19/3$:

$$P_{U|B}(u) = \begin{cases} 3/19, & u=0; \\ 6/19, & u=1; \\ 6/19, & u=2; \\ 4/19, & u=3. \end{cases}$$

Problem 8 *6 points*

Prove or **disprove** each of the following statements. (To "prove," give a clear and convincing argument. To "disprove," provide a counterexample and an explanation of why it is a counterexample.) The two parts are completely separate.

(a) If A, B, and C are events such that

$$\mathbb{P}[A \,|\, C] > \mathbb{P}[B \,|\, C] \quad \text{and} \quad \mathbb{P}[A \,|\, C^c] > \mathbb{P}[B \,|\, C^c]$$

then

$$\mathbb{P}[A] > \mathbb{P}[B].$$

(You may assume $0 < \mathbb{P}[C] < 1$.)

Solution:

The statement is true. Using the Law of Total Probability,

$$\mathbb{P}[A] = \mathbb{P}[A \,|\, C]\,\mathbb{P}[C] + \mathbb{P}[A \,|\, C^c]\,\mathbb{P}[C^c],$$
$$\mathbb{P}[B] = \mathbb{P}[B \,|\, C]\,\mathbb{P}[C] + \mathbb{P}[B \,|\, C^c]\,\mathbb{P}[C^c].$$

Subtracting the second from the first,

$$\mathbb{P}[A] - \mathbb{P}[B] = (\mathbb{P}[A \,|\, C] - \mathbb{P}[B \,|\, C])\,\mathbb{P}[C] + (\mathbb{P}[A \,|\, C^c] - \mathbb{P}[B \,|\, C^c])\,\mathbb{P}[C^c],$$

and this must be positive because each of the two terms is the product of two positive factors.

(b) If A, B, and C are events such that

$$\mathbb{P}[A \,|\, B] > \mathbb{P}[A] \quad \text{and} \quad \mathbb{P}[B \,|\, C] > \mathbb{P}[B],$$

then

$$\mathbb{P}[A \,|\, C] > \mathbb{P}[A].$$

(You may assume $\mathbb{P}[B] > 0$ and $\mathbb{P}[C] > 0$.)

Solution:

The statement is tantalizing but false. For example, let the sample space be the unit interval with the probability assigned to an event equal to its length, and let $A = [\frac{1}{16}, \frac{1}{4}]$, $B = [\frac{1}{8}, \frac{1}{2}]$, and $C = [\frac{1}{4}, \frac{9}{16}]$. Then

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{1/8}{3/8} = \frac{1}{3} > \frac{3}{16} = \mathbb{P}[A]$$

and

$$\mathbb{P}[B \mid C] = \frac{\mathbb{P}[B \cap C]}{\mathbb{P}[C]} = \frac{1/4}{5/16} = \frac{4}{5} > \frac{3}{8} = \mathbb{P}[B].$$

However $\mathbb{P}[A \mid C] = 0$, which is less than $\mathbb{P}[A] = \frac{3}{16}$.