Exam 1

First Name	Last Name	UID
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ____________________

Partial Credit: There will be partial credit for solution attempts even if not all the mathematical manipulations are completed correctly. It is advisable to attempt every problem.

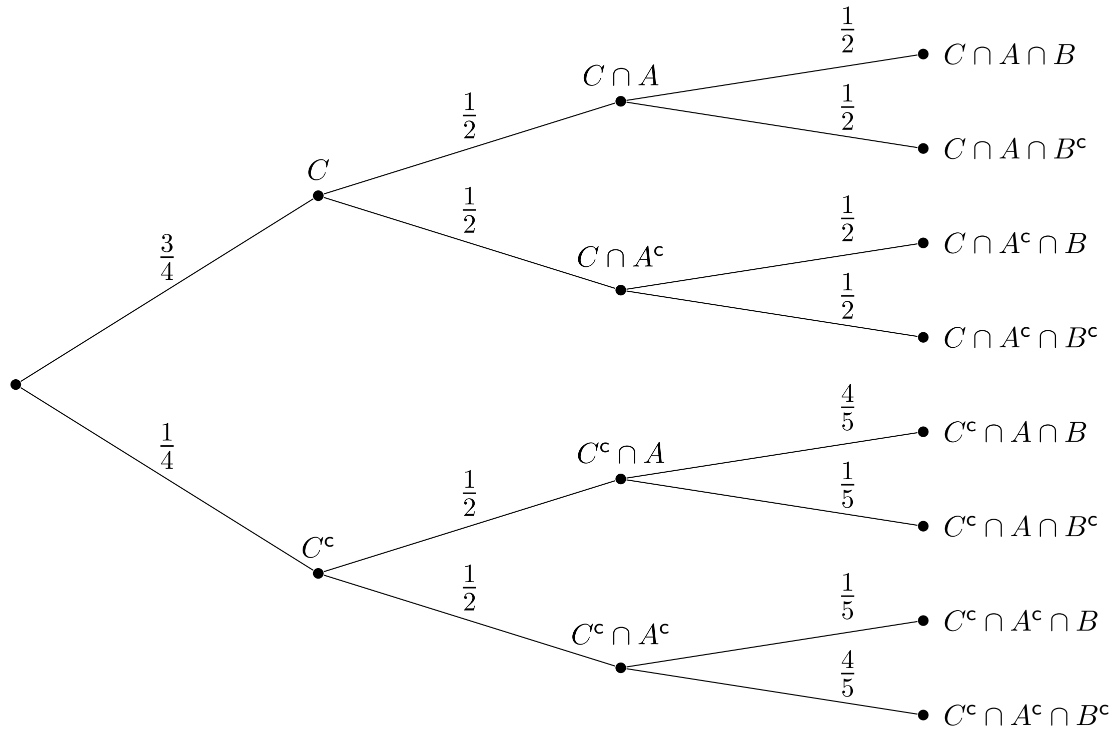
- You have exactly **2 hours** to complete this exam.
- **No devices are allowed**, including no phones and no calculators. No form of collaboration is allowed.
- You can use the provided formula sheet handout – no other materials are allowed.
- All work to be graded must be included in this document. Submit no extra sheets.
- Answers can be left in terms of sums without having to add or multiply all the terms, unless otherwise specified in the problem. The formula sheet includes an $\binom{n}{k}$ table for $n = 0, 1, \ldots, 8$ so you are expected to plug in these values when needed.
- There are 8 problems in total, with different point values. Problem 6 is **conceptually harder**. Don't get stuck in any one true/false question (Problems 7, 8).

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		12	Problem 5		16
Problem 2		16	Problem 6		12
Problem 3		16	Problem 7		6
Problem 4		16	Problem 8		6
			Total		100

Problem 1 *12 points*

The below probability tree describes the conditional probabilities involving three events A, B, C:



(a) Calculate $\mathbb{P}[A \cap C^c]$ and $\mathbb{P}[A \cap B \cap C^c]$ (4 pts)

(b) Calculate $\mathbb{P}[B \cap C^c]$ (4 pts)

(c) Determine if events A and B are conditionally independent given event C^c. Justify your answer with calculations. (4 pts)

Problem 2 *16 points*

You have ten coins: one penny, two nickels, three dimes, and four quarters. You randomly select one coin out of the ten, and then randomly select another coin out of the remaining nine. During each selection, all available coins are equally likely to be selected.

(a) Calculate the probability that both coins that you select are dimes. (4 pts)

(b) Calculate the probability that at least one coin that you select is a dime. (4 pts)

(c) Calculate the probability that the second coin is a dime, given that the first coin is a nickel. (4 pts)

(d) Calculate the probability that the first coin is a nickel, given that the second coin is a dime. (4 pts)

Problem 3 *16 points*

You and your friend play ten games of chess. Each game results in a win, loss, or draw. For any given game, the probability that you win is 0.3, and the probability that you lose is 0.2. The outcomes of the ten games are all independent of each other.

(a) Let W be the total number of games that you win. Identify the family of PMFs to which the PMF of W belongs and its parameters. Be as specific as possible (e.g., if you think W is Poisson(λ), you should state the value of λ). (4 pts)

(b) Let M be the total number of decisive games (i.e., wins or losses, but not draws). Calculate $\mathsf{Var}[M]$. (4 pts)

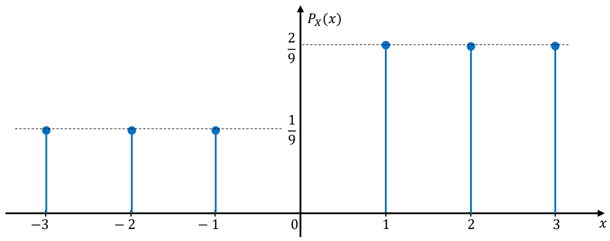
(c) A win counts as 1 point; a draw, 0.5 points; a loss, 0 points. Let S be your final score. Calculate $\mathbb{E}[S]$. (4 pts)

(d) You are distracted and end up losing all ten games. Your friend agrees to keep playing, but only until you win one game. Let N be the total number of games played till you win (including the original ten losses and the final win). Calculate $\mathbb{E}[N]$. (4 pts)

Problem 4 *16 points*

Let X have the probability mass function with case-by-case formula below and plot shown on the right.

$$P_X(x) = \begin{cases} 1/9, & x = -3, -2, -1; \\ 2/9, & x = 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$



(a) Calculate $\mathbb{E}[X]$ and $\mathsf{Var}[X]$. (4 pts)

(b) Let $Y = \begin{cases} 1 & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Compute $\mathsf{Var}[Y]$. (4 pts)

(c) Let B be the event that $|X| < 2$. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula. (4 pts)

(d) Compute $\mathsf{Var}[X \mid B]$, the variance of the random variable $X|B$ distributed according to the PMF $P_{X|B}(x)$ in the previous part. (4 pts)

Problem 5 *16 points*

We want to develop a probability model based on the results of two EK381 entrance surveys, summarized in the table on the right. The sample space Ω is all possible Engineering students at BU who register for EK381 in either Fall or, if not, in Spring of one academic year (not limited to these two surveys). We assume that students do not register in both semesters.

Registrations in EK381
Fall 2024/Spring 2025

	Semester		
Major	Fall	Spring	Total
BME	96	33	129
CE	46	18	64
EE	19	36	55
ME	54	94	148
Totals	215	181	396

Let S be a random variable which is 1 if a random student drawn from Ω takes EK381 in the Fall, and 0 if that student takes EK381 in the Spring and let events B, E, C, and M represent whether the student is a BME, EE, CE, or ME major, respectively.

(a) To which family does the PMF of S belong? Estimate its parameters using data in the table. (4 pts)

(b) To which family does the conditional PMF of S given B, i.e., $P_{S|B}(x)$, belong? Estimate its parameters using data in the table. (4 pts)

(c) If an instructor walks up to a random person in this exam room, what is the probability that the student is an ME major? Express this as a conditional probability and estimate its value using the table. (4 pts)

(d) Suppose we model the number of Fall ME students as $\text{Binomial}(150, \frac{50}{150})$ and suppose $\mathbb{P}[100 \text{ Fall ME students}] = 2^m \times \mathbb{P}[50 \text{ Fall ME students}]$. Calculate m and comment on the likelihood of a flipped enrollment (100 ME students in the Fall versus 50). Note: $\binom{n}{k} = \binom{n}{n-k}$. (4 pts)

Problem 6 *12 points*

To improve sales, a cereal company places one coupon inside each box of cereal and offers a small prize to anyone who collects and submits 5 coupons all of different colors (no two have the same color). The coupon colors are assigned independently to each box and they are equally likely to be either red, green, blue, white, or orange (5 different colors). You have an unlimited number of cereal boxes to choose from.

(a) After buying the first box of cereal, what is the expected number of *additional* boxes you would need to buy to find a coupon of a different color than the first? (4 pts)

(b) After you collect coupons of two different colors, what is the expected number of *additional* boxes you would need to buy to find a coupon of a third color? (4 pts)

(c) What is the expected number of *total* boxes you would need to buy to win a prize? (4 pts)

Problem 7 *6 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If X is Bernoulli(p) and $Y = 2X - 1$, then $\mathbb{E}[Y^2] = 1$. (2 pts)

(b) If X has $\mathbb{E}[X] = 5$ and standard deviation 5, then X could be a Poisson RV. (2 pts)

(c) If X is Uniform(a, b) and $P_X(0) = 0.5$, then $b - a = 2$. (2 pts)

Problem 8 *6 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that the sample space is denoted by Ω, that A, B, and C are events with $\mathbb{P}[A], \mathbb{P}[B], \mathbb{P}[C] \in (0, 1]$.

(a) If $0 < \mathbb{P}[B] < 1$, then $\mathbb{P}[A|B] + \mathbb{P}[A|B^c] = 1$ for any A. (2 pts)

(b) If $\mathbb{P}[A] = 0.8$ and $\mathbb{P}[B|A] = 0.5$, then A and $(A \cap B)$ must be independent. (2 pts)

(c) If $\mathbb{P}[A] = 0.8$ and $\mathbb{P}[B|A] = 0.5$, then $\mathbb{P}[A] > \mathbb{P}[B]$. (2 pts)