ENG EK381 Probability, Statistics and Data Science for Engineers — Spring 2024
 — February 14, 2024

Exam 1

Last Name	First Name	Student ID #
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ______________________________

Partial Credit: There will be partial credit for solution attempts even if not all the mathematical manipulations are completed correctly. It is advisable to attempt every problem.

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed** – no phones and no calculators. No form of collaboration is allowed.
- You can use the provided formula sheet handout – no other materials are allowed.
- All work to be graded must be included in this document. Submit no extra sheets. The blank page at the end can be used for scratch work, but it must remain attached. The final answer to each part must be written back in the page with the problem statement.
- Answers can be left in terms of sums without having to add or multiply all the terms, unless otherwise specified in the problem. Combinations and factorials can be left as part of answers without simplification, unless the combination values are provided for the problem.
- There are 8 problems in total, with different point values. Don't get bogged down with any one true/false question. The last problem is **worth little, but is conceptually harder**; attempt all other problems before this one.

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		10	Problem 5		12
Problem 2		10	Problem 6		16
Problem 3		16	Problem 7		16
Problem 4		16	Problem 8		4
			Total		100

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$. The sample space is denoted by Ω.

(a) Suppose $\mathbb{P}[A] = 0.5, \mathbb{P}[B] = 0.5$ and and $\mathbb{P}[A \cup B] = 1$. Then, A and B cannot be independent.

(b) $\mathbb{P}[A|B \cap C] \leq \min\{\mathbb{P}[A|B], \mathbb{P}[A|C]\}$ (equivalently, $\mathbb{P}[A|B \cap C] \leq \mathbb{P}[A|B]$ and $\mathbb{P}[A|B \cap C] \leq \mathbb{P}[A|C]$).

(c) $\mathbb{P}(A \cap B) + \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

(d) If A, B, C are independent, then $\mathbb{P}[A \cup B \cup C] = 1 - \mathbb{P}[A]\,\mathbb{P}[B]\,\mathbb{P}[C]$.

(e) If A, B, and C are events such that $\mathbb{P}[A \mid B] > \mathbb{P}[A]$ and $\mathbb{P}[B \mid C] > \mathbb{P}[B]$, then $\mathbb{P}[A \mid C] > \mathbb{P}[A]$.

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If for some constant c we have $\mathbb{P}[\{X > c\}] = 1$, then $\mathbb{E}[X] > c$.

(b) If $F_X(0) > 1/2$, then $\mathbb{E}[X] \leq 0$.

(c) If $F_X(3) = F_X(2)$, then $P_X(3) = 0$.

(d) If X is a Geometric($\frac{1}{4}$) random variable, then $\mathbb{E}[\ln(X)] > 0$.

(e) If $Y = X + 2$, then $F_Y(a) = F_X(a-2)$.

Problem 3 Complete the following quick calculations. *16 points*

(a) Let X be Discrete Uniform $(1, b)$, and let $\mathbb{E}[X] = 5$. Compute b and $\mathsf{Var}[X]$.

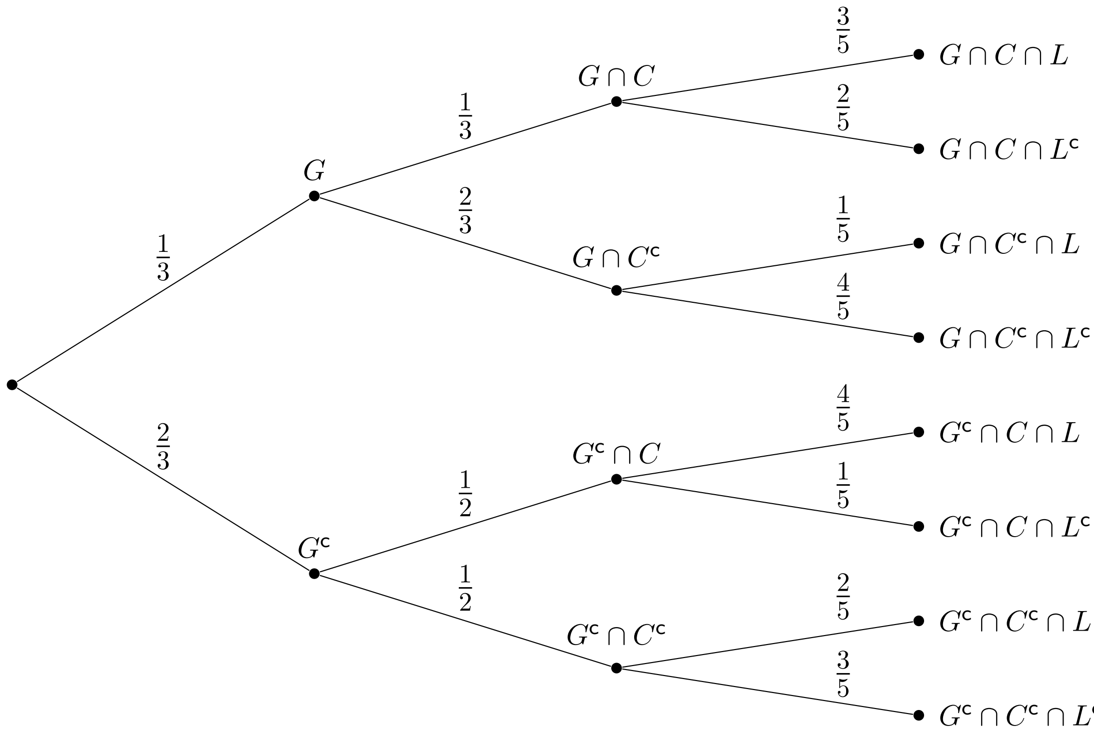
(b) Let X be Poisson (3). Calculate $\mathbb{E}[X+1]$ and $\mathsf{Var}[3-2X]$.

(c) Let X be a Bernoulli $(\frac{1}{3})$ random variable. Compute $\mathbb{E}[3-2X]$ and $\mathbb{E}[2^X]$.

(d) Let A and B be events with $\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[A \cap B] = \dfrac{3}{8}$ and $\mathbb{P}[A^{\mathsf{c}} \cap B] = \dfrac{1}{8}$.
Calculate $\mathbb{P}[B]$ and $\mathbb{P}[A|B^c]$.

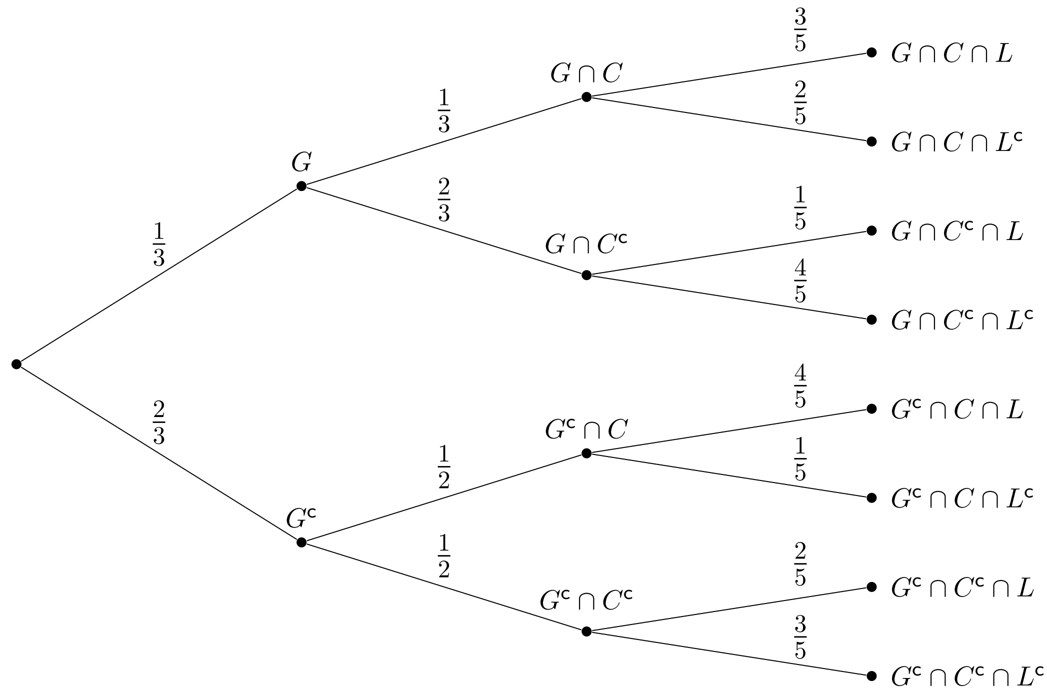
Problem 4 *16 points*

You always take the same bus to school and have built a probability model to predict when you will be late. Specifically, you have made the following conditional probability tree where G is the event that the weather is good, C is the event that the bus is crowded, and L the event that you are late to class.



(a) What is the probability that the weather is good and you are late to class?

(b) Given that the weather is good, what is the probability of being late to class?



(c) What is the probability of being late to class?

(d) Given that you are late to class, what is the probability that the bus was crowded?

Problem 5 *12 points*

Consider the following simple lottery game. There are ten balls total of which five are red and numbered $1, 2, 3, 4, 5$. The remaining five are blue and also numbered $1, 2, 3, 4, 5$. Three balls are simultaneously pulled out from a hat, and depending on what they are, you might win a prize. Useful facts: $\binom{10}{3} = 120$; $\binom{5}{3} = 10$.

(a) You win a teddy bear if all three balls are the same color. What is the probability of winning a teddy bear?

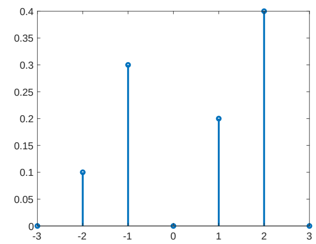
(b) You decide to play the same lottery game for 4 days in a row (and each day's outcome is independent of the others). What is the probability that you win *at least one* teddy bear?

(c) In the same lottery, you can also win a Saint Seiya figurine if the three balls have consecutive numbers (such as red 2, blue 3, and blue 4) regardless of color (or order). What is the probability of winning a Saint Seiya?

Problem 6 *16 points*

Let X have the probability mass function below, with plot shown on the right.

$$P_X(x) = \begin{cases} 1/10, & x = -2; \\ 3/10, & x = -1; \\ 2/10, & x = 1; \\ 4/10, & x = 2; \\ 0, & \text{otherwise.} \end{cases}$$



(a) Calculate $\mathbb{E}[X]$ and $\mathsf{Var}[X]$.

(b) Let B be the event that $|X| < 2$. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula.

(c) Compute $\mathsf{Var}[X \mid B]$, the variance of the random variable $X|B$ distributed according to the PMF $P_{X|B}(x)$ in the previous part.

(d) Compute $\mathbb{E}\left[\frac{1}{X^3} \mid B\right]$.

Problem 7 *16 points*

A service center receives a random number X of service requests per day, which we model as a Binomial(5,1/4) random variable. The number of service requests is independent from day to day. Useful formulas: $\binom{5}{2} = \binom{5}{3} = 10$; $\binom{5}{1} = \binom{5}{4} = 1$. Your answers should not contain combinations.

(a) What is the probability of getting either two or three service requests in a day?

(b) Given that, on a given day, you know the center received no more than two requests, what is the probability it has only one request?

(c) What is the probability that, on a given day, the service center receives the maximum number of five requests?

(d) What is the expected number of days until the service center receives five requests in a day?

Problem 8 *3 points*

A black dog and a brown dog are going to have a litter of puppies. The number of puppies in the litter is a Discrete Uniform(2,6) random variable. For any size litter, the probability that a puppy will be black is $\frac{3}{4}$ and brown $\frac{1}{4}$, and the colors of the puppies are independent across the litter.

(a) (1 pt) What is the expected number of black puppies?

(b) (3 pts) What is the probability that we get at least 4 black puppies?