Exam 1

Last Name	First Name	Student ID #
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ______________________________

Partial Credit: The most important issue is knowing how to approach a particular problem. Therefore, there will be partial credit for good solution outlines even if not all the mathematical manipulations are completed correctly. Be sure to attempt every problem!

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed** - including no phones and no calculators.
- You can use the provided formula sheet handout - no extra materials are allowed.
- No form of collaboration is allowed.
- All work to be graded must be included in this document. Submit no extra sheets.
- There are 8 problems in total, with different point values. The last problem is *worth little, but is conceptually harder*. Make sure you attempt all other problems before this one.

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		10	Problem 5		15
Problem 2		10	Problem 6		15
Problem 3		16	Problem 7		15
Problem 4		15	Problem 8		4
			Total		100

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$.

(a) $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)(1 - \mathbb{P}(B^c))$

(b) If A and B are independent, $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]\mathbb{P}[A^c]$.

(c) If $A \subset B$, then $\mathbb{P}[A|C] \leq \mathbb{P}[B|C]$.

(d) If $A \cup B = \Omega$, where Ω is the sample space, then $\mathbb{P}[A] + \mathbb{P}[B] = 1$.

(e) $\mathbb{P}[A \mid B] \geq \mathbb{P}[A]$.

Problem 2 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Briefly explain the reasoning behind your answer for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $F_X(1) = F_X(3)$, then $P_X(2) = 0$.

(b) $\mathsf{Var}(3 - X) = \mathsf{Var}(X)$.

(c) For any random variable X, $\mathbb{E}[X^3] = (\mathbb{E}[X])^3$.

(d) Let X be a Binomial$(10, 1/4)$ random variable and Y be a Binomial$(5, 1/4)$ random variable. Then, $X + Y$ is a Binomial$(15, 1/4)$ random variable.

(e) If $x \in B$, and $\mathbb{P}[B] > 0$, then $P_{X|B}(x) \geq P_X(x)$.

Problem 3 Please complete the following quick calculations. *16 points*

(a) Suppose that out of the students in a class, 60% are geniuses (event G), 70% love chocolate (event C), and 40% fall into both categories (event $G \cap C$). Determine the probability that a randomly selected student is neither a genius nor a chocolate lover ($\mathbb{P}[(G \cup C)^c)]$, and the probability that a genius loves chocolate ($\mathbb{P}[C|G]$).

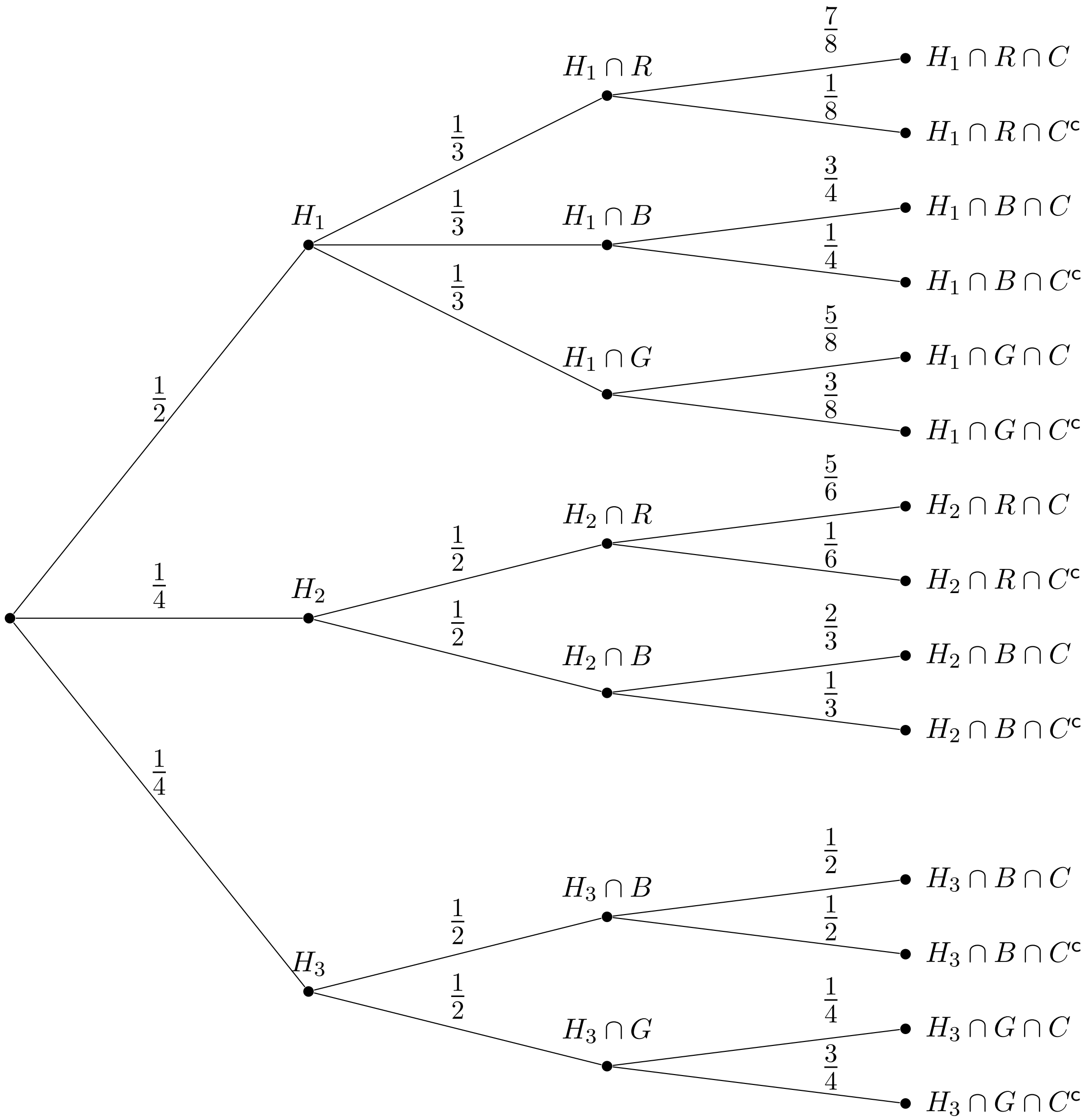
(b) Let X be Binomial$(7, \frac{1}{2})$. Calculate $\mathbb{P}[1 \leq X \leq 6]$ and $\mathbb{P}\big[X = 2 \,\big|\, 1 \leq X \leq 6\big]$. Useful facts: $2^7 = 128$; $\binom{7}{1} = \binom{7}{6} = 7$; $\binom{7}{2} = \binom{7}{5} = 21$; $\binom{7}{3} = \binom{7}{4} = 35$.

(c) Let X be Discrete Uniform$(1, 37)$. Let B the event that X is strictly greater than 4 *and* strictly less than 8. Calculate $\mathbb{E}[X|B]$ and $\mathbb{E}[X^2|B]$.

(d) Let X be Poisson(2). Calculate $\mathbb{E}[3X - 1]$ and $\mathbb{E}[3 - X^2]$.

Problem 4 *15 points*

You are playing a simple game where you first select one of three hats, and then reach into the hat and grab a box, which is either red, green, or blue, and then finally open the box to see if there is a chocolate inside. Specifically, let H_i be the event that you choose the i^{th} hat, R the event that you pick a red box, G the event that you pick a green box, B the event that you pick a blue box, and C the event that you find a chocolate. A tree diagram with conditional probabilities on the branches is shown below.



(a) What is the probability that you choose a blue box?

(b) What is the probability that you choose a green box and do not find a chocolate?

(c) Given that you choose a green box and do not find a chocolate, what is the probability that you selected the third hat?

(d) Given that you choose a blue box, what is the probability that you find a chocolate?

(e) Given that you select the first hat and find a chocolate, what is the probability you choose a blue box?

Problem 5 *15 points*

The number of geese X you see on the Charles River Path on a stroll is modeled by a Poisson (λ) random variable. (Throughout the problem, you might encounter terms that include $e^{-\lambda}$ for some λ. You can leave these as is, since you should not use a calculator.)

(a) Say that you learn that $\mathbb{E}[5X] = 10$. Select the parameter λ based on this fact, and set it equal to this value for the remainder of the problem.

(b) Is it more likely to observe 1 goose or 3 geese on your stroll?

(c) Say that you learn that the number of geese you will observe on cloudy days is at most 3. Given that it is cloudy, what is the average number of geese you observe?

(d) Given that it is cloudy, what is the probability you do not see any geese?

(e) On a sunny day, someone tells you there is at least one goose on the path. Given this information, what is the probability that there are at least two geese on the path?

Problem 6 *16 points*

Audi's production facility for Audi S4's in Ingolstadt (Germany) works around the clock and produces one Audi S4 per hour. Every car produced is tested and is released only if found performing according to the specifications. Otherwise, defective cars are sent to a special unit in Györ (Hungary) to fix them. Suppose that the probability of a car passing the test (and thus released) is $\frac{3}{4}$, and passing the test is independent at the end of each hour. Note that they have very stringent standards for releasing a luxury car.

(a) Suppose that a car is released from Ingolstadt at 4:00pm today. What is the probability that they will release the next car at 7:00pm today?

(b) What is the probability that they release from Ingolstadt 2 cars in a 5 hour period? Useful facts: $\binom{5}{1} = \binom{5}{4} = 5; \binom{5}{2} = \binom{5}{3} = 10\,,\; 4^5 = 1024\,.$

(c) Suppose every defective car sent to Györ (Hungary) for fixing will cost Audi an additional \$10,000 in transportation and labor. What is the expected cost per 24-hour day that Audi incurs for fixing defective cars?

(d) What is the expected number of hours that Audi requires to manufacture 300 non-defective cars?

Problem 7 *15 points*

Consider the following game. There are eight balls total of which 4 are red and numbered $1, 2, 3, 4$. The remaining four are blue and also numbered $1, 2, 3, 4$. Three balls are simultaneously pulled out from a hat, and depending on what they are, you might win a prize. Useful facts: $\binom{8}{1} = \binom{8}{7} = 8$; $\binom{8}{2} = \binom{8}{6} = 28$; $\binom{8}{3} = \binom{8}{5} = 56$; $\binom{8}{4} = 70$.

(a) What is the total number of ways that you can draw three balls out of eight?

(b) You win a Naruto action figure if all three balls that you draw are the same color. What is the probability of winning a Naruto?

(c) You win a Saint Seiya action figure if two of the balls have the same number. What is the probability that you win a Saint Seiya action figure?

(d) What is the probability that you win either prize?

Problem 8 *4 points*

Two players N (Nadal) and D (Djokovic) are playing a tennis game on clay at the French Open. Player N is serving in this game. Player N wins each point with probability $\frac{2}{3}$ while player D wins with probability $\frac{1}{3}$. Each point may be regarded as an independent trial. Let T denote the event that N and D each win 3 of the first 6 points played. The score is then deuce, meaning tied.

(a) What is $\mathbb{P}(T)$, the probability that deuce was reached? You can leave the answer in terms of combinations and powers. (Hint: need 3 wins for each player in the first six points played.) Useful facts: $\binom{6}{1} = 6; \binom{6}{2} = \binom{6}{4} = 15; \binom{6}{3} = 20;\ 2^6 = 64; 3^6 = 729$.

(b) Once the score reaches deuce, a player must win two more points than the opponent in order to win the game. Given that the score is deuce (Event T occurs), what is the (conditional) probability that N wins the next two points (and hence the game)?

(c) Let X denote the number of points played after the players reach the first deuce, (event T), until someone wins the game by being ahead 2 points. What is the expected value of X given T, the event that deuce was reached after 6 points?.