Exam 1

Last Name	First Name	Student ID #
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Honor Code: This exam represents only my own work. I did not give or receive help.

Signature: ____________________

Partial Credit: There will be partial credit for good solution outlines even if not all the mathematical manipulations are completed correctly. It is advisable to attempt every problem.

- You have exactly **2 hours** to complete this exam.
- **No devices are allowed** – no phones and no calculators.
- No form of collaboration is allowed.
- You can use the provided formula sheet handout – no other materials are allowed.
- Your exam and formula sheet must be kept within your desk space. Unoccupied seats between students are to remain empty.
- All work to be graded must be included in this document. Submit no extra sheets. The blank page at the end can be used for scratch work, but it must remain attached.
- There are 8 problems in total, with different point values. Don't get bogged down with any one true/false question. The last problem is *worth little, but is conceptually harder*; attempt all other problems before this one.

*** Good Luck! ***

Problem	Points earned	out of	Problem	Points earned	out of
Problem 1		10	Problem 5		12
Problem 2		8	Problem 6		16
Problem 3		16	Problem 7		16
Problem 4		16	Problem 8		6
			Total		100

Problem 1 *10 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that A, B, and C are events with $\mathbb{P}[A] > 0$, $\mathbb{P}[B] > 0$, and $\mathbb{P}[C] > 0$. The sample space is denoted by Ω.

(a) If $A \cup B = \Omega$, then $\mathbb{P}[A] + \mathbb{P}[B] = 1$.

(b) If $\mathbb{P}[A \mid B] < \mathbb{P}[A]$, then $\mathbb{P}[A \cap B] > \mathbb{P}[B]$.

(c) $\mathbb{P}[A \cap B^c] = \mathbb{P}[A] - \mathbb{P}[A \cap B]$

(d) $\mathbb{P}[A^c \cap B^c] = 1 - \mathbb{P}[A] - \mathbb{P}[B]$.

(e) $\mathbb{P}[A] + \mathbb{P}[A^c] + \mathbb{P}[B] > 1$.

Problem 2 *8 points*

For each of the following parts, indicate whether the statement is always true or it can be false by clearly writing "True" or "False." Full credit will be given for selecting the correct logical value (even with no explanation). You are welcome to briefly explain your reasoning for partial credit (in case your choice is wrong). Diagrams are welcome. Throughout the problem, you may assume that X is a discrete random variable with PMF $P_X(x)$ and CDF $F_X(x)$.

(a) If $\mathbb{E}[(X+3)^2] = \mathbb{E}[X^2] + 9$, then $\mathbb{E}[X] = 0$.

(b) For some values of the constant c, $\mathsf{Var}[cX] < \mathsf{Var}[X]$.

(c) If for some constant c we have $\mathbb{P}[X < c] > 1/2$, then $\mathbb{E}[X] < c$.

(d) If $F_X(a) > F_X(b)$, then $a > b$.

Problem 3 Complete the following quick calculations. *16 points*

(a) Consider independent events A, B, and C satisfying $\mathbb{P}[A] = 1/3$, $\mathbb{P}[B] = 1/2$, $\mathbb{P}[C] = 2/3$. Calculate $\mathbb{P}[A \cap C^c]$ and $\mathbb{P}[A \cup (B \cap C)]$.

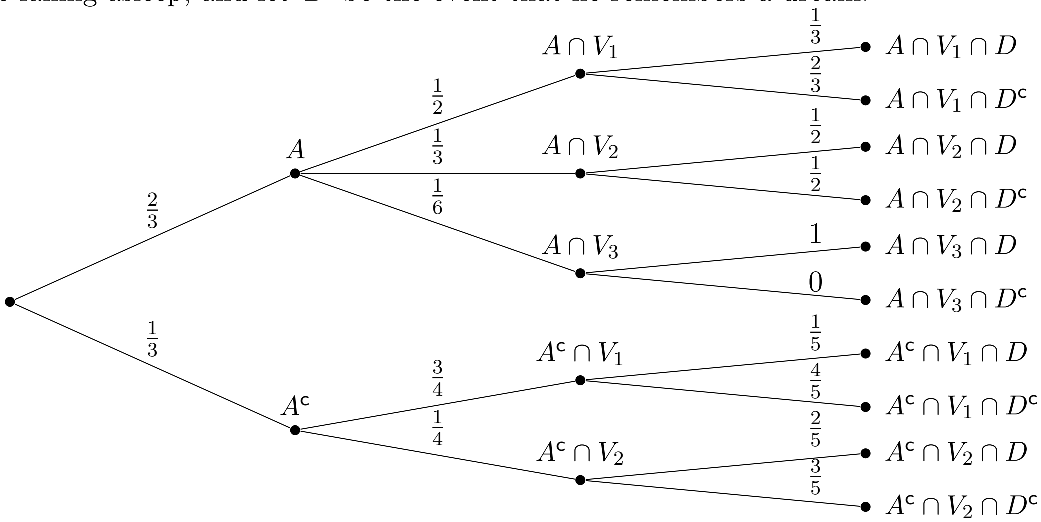
(b) Let A and B be events with $\mathbb{P}[A \mid B] = 1/3$, $\mathbb{P}[A \mid B^c] = 1/6$ and $\mathbb{P}[B^c] = 3/4$. Calculate $\mathbb{P}[A \cap B]$ and $\mathbb{P}[B \mid A]$.

(c) Let X be Binomial $(4, \frac{1}{3})$. Calculate $\mathbb{P}[X \geq 1]$ and $\mathbb{E}[4X - 2]$.

(d) Let X be Discrete Uniform $(-1, 2)$. Calculate $\mathbb{E}[X^2]$ and $\mathbb{E}[X^2 \mid X < 2]$.

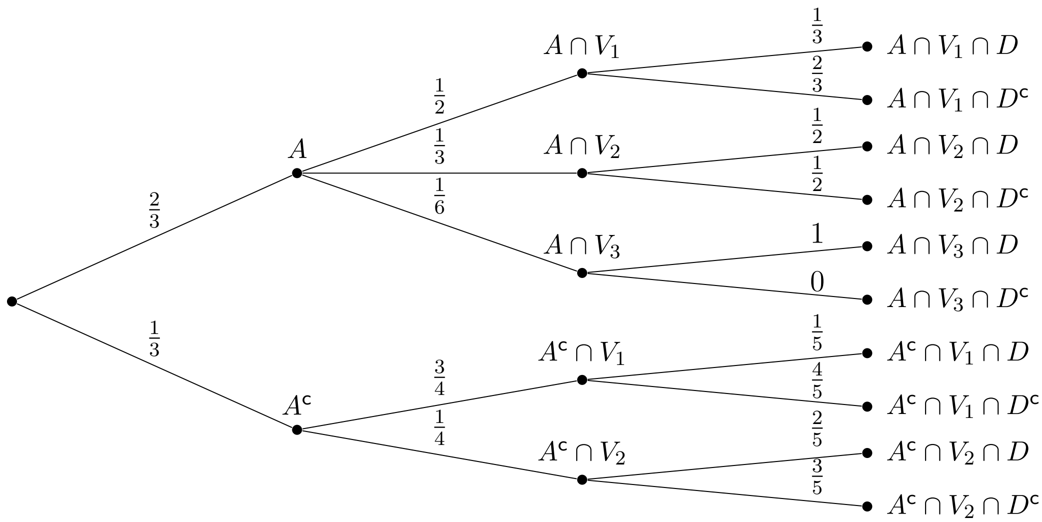
Problem 4 *16 points*

Vincent uses the tree diagram below for a model of whether he wakes up remembering a dream. Let A be the event that he drinks absinthe before going to bed, let V_i be the event that he plays i games of Wordle in bed before falling asleep, and let D be the event that he remembers a dream.



(a) What is the probability that Vincent plays two games of Wordle before falling asleep?

(b) Given that Vincent drank absinthe before going to bed, what is the probability that he remembers a dream?



(c) Given that Vincent remembers a dream, what is the probability that he drank absinthe before going to bed?

(d) Given that Vincent played two games of Wordle and remembers a dream, what is the probability that he drank absinthe?

Problem 5 *12 points*

Each part of this problem uses a well-shuffled standard 52-card deck: four suits (spades, hearts, diamonds, and clubs), 13 cards of each suit (numbers from 2 to 10, Jack, Queen, King, and Ace). Remember that expressions with factorials and binomial coefficients are fine in your final answers.

(a) What is the probability that a 3-card hand is all spades?

(b) What is the probability that a 6-card hand has 4 cards of one suit and 2 cards of a second suit?

(c) Given that a 6-card hand contains at least two suits, what is the probability that the 6 cards are 4 of one suit and 2 of a second suit?

Problem 6 *16 points*

Let X have the probability mass function

$$P_X(x) = \begin{cases} 1/8, & x = -2; \\ 1/2, & x = -1; \\ 1/8, & x = 1; \\ 1/4, & x = 2; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Make clearly labeled sketches of the PMF $P_X(x)$ and CDF $F_X(x)$.

(b) Calculate $\mathbb{E}[X]$ and $\mathsf{Var}[X]$.

(c) Let B be the event that $X < 2$. Determine the conditional PMF $P_{X|B}(x)$ and write it down as a case-by-case formula.

(d) Compute $\mathbb{E}\left[\frac{1}{X} \mid B\right]$.

Problem 7 *16 points*

The number of spam text messages you receive each day has the Poisson(2) distribution, and the number of these messages is independent from day to day.

(a) Let X be the number of days in a week (7 days) that you receive no spam text messages. What is $\mathsf{Var}[X]$?

(b) What is the probability that in one week, there are exactly 4 days on which you receive exactly 3 spam text messages?

(c) Let Y be the number of days until the first on which you receive 2 or more spam text messages. What is $\mathbb{E}[Y]$?

(d) Given that you have received at most 3 spam text messages in a day, what is the conditional PMF of the number of spam text messages received?

Problem 8 *6 points*

Prove or **disprove** each of the following statements. (To "prove," give a clear and convincing argument. To "disprove," provide a counterexample and an explanation of why it is a counterexample.) The two parts are completely separate.

(a) If A, B, and C are events such that

$$\mathbb{P}[A \mid C] > \mathbb{P}[B \mid C] \quad \text{and} \quad \mathbb{P}[A \mid C^c] > \mathbb{P}[B \mid C^c]$$

then

$$\mathbb{P}[A] > \mathbb{P}[B].$$

(You may assume $0 < \mathbb{P}[C] < 1$.)

(b) If A, B, and C are events such that

$$\mathbb{P}[A \mid B] > \mathbb{P}[A] \quad \text{and} \quad \mathbb{P}[B \mid C] > \mathbb{P}[B],$$

then

$$\mathbb{P}[A \mid C] > \mathbb{P}[A].$$

(You may assume $\mathbb{P}[B] > 0$ and $\mathbb{P}[C] > 0$.)